10.5 – Arc length for Parametric Curves

Recall: Formula for finding the arc length of a curve in rectangular form:

$$L(c) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$y = f(x)$

Formula for finding the arc length of a curve in polar form:

$$L(c) = \int_\alpha^\beta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta$$

$r(\theta) =$ ______

New: Formula for finding the arc length of a curve in parametric form:

$$L(c) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, \underline{dt}$$

t values

Velocity and Speed (magnitude of velocity)

If the position of a particle at time t is given by

$$s(t) = (x(t), y(t))$$

then the velocity is given by

$$v(t) = (x'(t), y'(t))$$

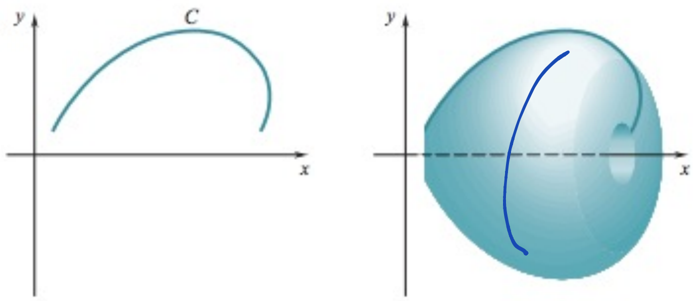
and the speed is given by

$$\text{speed} = \|v(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

Let C be a curve in the upper half-plane (see figure). The curve can meet the x-axis, but only at a finite number of points. We will assume that C is parametrized by a pair of continuously differentiable functions

$$x = x(t), \quad y = y(t), \quad t \in [c, d].$$

Furthermore, we will assume that C is *simple*: no two values of t between c and d give rise to the same point of C; that is, the curve does not intersect itself.



If we revolve C about the x-axis, we obtain a surface of revolution. The area of that surface is given by the formula

$$\boxed{A = \int_c^d 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.}$$

PRINTABLE VERSION

Quiz 26 – 10.5

You scored 0 out of 100

Question 1

You did not answer the question.

$x'(t) = 2 \quad y'(t) = 7$

Find the length of the curve of $x(t) = 2\,t,\ y(t) = 7\,t - 4,$ for $t \in [0, 6]$.

$$\int_0^6 \sqrt{2^2 + 7^2}\, dt = \int_0^6 \sqrt{53}\, dt$$

$$= \sqrt{53}\, t \Big|_0^6 = 6\sqrt{53}$$

a) $6\sqrt{53}$

b) $9\sqrt{53}$

c) $12\sqrt{53}$

d) $18\sqrt{53}$

e) $3\sqrt{53}$

Question 2

You did not answer the question.

Find the length of the curve of $x(t) = 2\,t^3,\ y(t) = 4\,t^2 + 4,$ for $t \in [0, 1]$.

$x'(t) = 6t^2 \quad y'(t) = 8t$

$$\int_0^1 \sqrt{36t^4 + 64t^2}\, dt = \frac{1}{72}\int_0^1 72t\, \sqrt{36t^2 + 64}\, dt$$

$u = 36t^2 + 64$

$du = 72t\, dt$

$$\frac{1}{72}\int_{64}^{100} \sqrt{u}\, du = \frac{1}{72} \cdot \frac{2}{3} u^{3/2} \Big|_{64}^{100}$$

$$\frac{1}{108}\left(100^{3/2} - 64^{3/2}\right)$$

$1000 - 512$

a) $\frac{61}{9}$

b) $\frac{61}{27}$

c) $\frac{122}{9}$

d) $\frac{244}{27}$

e) $\frac{122}{27}$

Question 3

You did not answer the question.

Find the length of the curve of $x(t) = 3\cos(4t)$, $y(t) = 3\sin(4t)$, for $t \in [0, 2]$.

$x'(t) = -12\sin(4t)$ $y'(t) = 12\cos(4t)$

$\int_0^2 \sqrt{144\sin^2(4t) + 144\cos^2(4t)}\, dt$

$\int_0^2 \sqrt{144(\sin^2(4t) + \cos^2(4t))}\, dt$

$\int_0^2 \sqrt{144}\, dt = \int_0^2 12\, dt = 12t\Big|_0^2$

a) 48

b) 24

c) 72

d) 96

e) 6

Question 4

You did not answer the question.

Which of the following integrals will find the length of the curve of $x(t) = 2t - 1$, $y(t) = t^2 + 3t$, for $t \in [2, 6]$?

$x'(t) = 2$ $y'(t) = 2t + 3$

a) $\int_2^6 \sqrt{4 + (2t+3)^2}\, dt$

b) $\int_2^6 \sqrt{(2t-1)^2 + \left(t^2 + 3t\right)^2}\, dt$

c) $\int_2^6 \sqrt{4 + \left(t^2 + 3t\right)^2}\, dt$

d) $\int_2^6 \sqrt{(2t-1)^2 + (2t+3)^2}\, dt$

e) $\int_2^6 \sqrt{(2t+3)^2 + 1}\, dt$

Question 5

You did not answer the question.

Which of the following integrals will find the length of the curve of

$x(t) = 3 - \cos(t),\ \ y(t) = 6\,t + \sin(t),\ \text{ for } t \in [3, 6]$?

a) $\displaystyle\int_3^6 \sqrt{2\,(\cos(t))^2 + 6\ \cos(t) + 45}\ \mathrm{d}t$

b) $\displaystyle\int_3^6 \sqrt{(\cos(t))^2 + 12\ \cos(t) + 37}\ \mathrm{d}t$

c) $\displaystyle\int_3^6 \sqrt{10 + 12\,t\sin(t) + 36\,t^2 - 6\ \cos(t)}\ \mathrm{d}t$

d) $\displaystyle\int_3^6 \sqrt{\left(\sin(t)\right)^2 + \left(6\,t + \sin(t)\right)^2}\ \mathrm{d}t$

e) $\displaystyle\int_3^6 \sqrt{37 + 12\ \cos(t)}\ \mathrm{d}t$

Question 6

You did not answer the question.

The equations $x(t) = t^2$ and $y(t) = t^3$ give the position of a particle at each time t from $t = 0$ to $t = 3$. Find the initial speed of the particle, the terminal speed, and the distance traveled.

(t^2, t^3) $\quad v(t) = (2t, 3t^2)$ $\quad \|v(t)\| = \sqrt{4t^2 + 9t^4}$

a) initial speed = 0, terminal speed = $2\sqrt{85}$; distance traveled = $\dfrac{85\sqrt{85}}{27} - \dfrac{8}{27}$

b) initial speed = 0, terminal speed = $\dfrac{9\sqrt{85}}{2}$; distance traveled = $\dfrac{85\sqrt{85}}{27} - \dfrac{8}{27}$

c) initial speed = 0, terminal speed = $3\sqrt{85}$; distance traveled = $\dfrac{85\sqrt{85}}{27} - \dfrac{8}{27}$

d) initial speed = 0, terminal speed = $6\sqrt{85}$; distance traveled = $\dfrac{170\sqrt{85}}{27} - \dfrac{16}{27}$

e) initial speed = 0, terminal speed = $\dfrac{3\sqrt{85}}{2}$; distance traveled = $\dfrac{85\sqrt{85}}{54} - \dfrac{4}{27}$

term. speed is $\|v(t)\|$ w/ $t=3$

$\sqrt{36 + 729}$

$\sqrt{765}$

$\displaystyle\int_0^3 \sqrt{4t^2 + 9t^4}\ dt = \int_0^3 t\sqrt{4 + 9t^2}\ dt$

(see #2)

Question 7

You did not answer the question.

$x'(t) = 3e^t \sin t + 3e^t \cos t = 3e^t(\sin t + \cos t)$ $y'(t) = 3e^t(\cos t - \sin t)$

The equations $x(t) = 3\,e^t \sin(t)$ and $y(t) = 3\,e^t \cos(t)$ give the position of a particle at each time t from $t = 0$ to $t = \pi$. Find the initial speed of the particle, the terminal speed, and the distance traveled.

$\sqrt{9e^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t) + 9e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t)}$

$3e^t\sqrt{1 + \cancel{2\sin t\cos t} + 1 \cancel{- 2\sin t\cos t}}$

a) initial speed = $6\sqrt{2}$, terminal speed = $6\sqrt{2}e^{\pi}$; distance traveled $=6\sqrt{2}(e^{\pi} - 1)$

b) initial speed = $2\sqrt{2}$, terminal speed = $2\sqrt{2}e^{\pi}$; distance traveled $=3\sqrt{2}(e^{\pi} - 1)$

c) initial speed = $\frac{9\sqrt{2}}{2}$, terminal speed = $9/2\sqrt{2}e^{\pi}$; distance traveled $=3\sqrt{2}(e^{\pi} - 1)$ Speed: $3e^t\sqrt{2}$

d) ✓ initial speed = $3\sqrt{2}$, terminal speed = $3\sqrt{2}e^{\pi}$; distance traveled $=3\sqrt{2}(e^{\pi} - 1)$ dist $= \int_0^{\pi} 3\sqrt{2}\, e^t\, dt$

e) initial speed = $\frac{3\sqrt{2}}{2}$, terminal speed = $3/2\sqrt{2}e^{\pi}$; distance traveled $=3/2\sqrt{2}(e^{\pi} - 1)$

Question 8

You did not answer the question.

Find the surface of revolution if the curve $x(t) = t^2 - 2$, $y(t) = 2t$, for $t \in [0, 4]$ is revolved around the x-axis.

$x'(t) = 2t$ $y'(t) = 2$

a) $\pi\left(\frac{68\sqrt{17}}{3} - \frac{4}{3}\right)$

$$\int_0^4 2\pi(2t)\sqrt{(2t)^2 + (2)^2}\, dt$$

b) $2\pi\left(\frac{68\sqrt{17}}{3} - \frac{4}{3}\right)$

c) $\left(\frac{68\sqrt{17}}{3} - \frac{4}{3}\right)$

$$\int_0^4 4\pi t\sqrt{4t^2 + 4}\, dt$$

d) $\pi\left(\frac{34\sqrt{17}}{3} - \frac{2}{3}\right)$

$$8\pi\int_0^4 t\sqrt{t^2+1}\, dt$$ $u = t^2 + 1$, $du = 2t\, dt$

e) $2\pi\left(\frac{34\sqrt{17}}{3} - \frac{2}{3}\right)$

$$4\pi\int_1^{17}\sqrt{u}\, du$$

Question 9

You did not answer the question.

Find the surface of revolution if the curve $x(t) = 3\,t + 2,\ y(t) = 5\,t,$ for $t \in [0, 4]$ is revolved around the x-axis.

$$\int_0^4 2\pi(5t)\sqrt{(3)^2 + (5)^2}\,dt$$

a) $160\sqrt{34}\pi$

b) $80\sqrt{34}$

c) $40\sqrt{34}\pi$

d) $80\sqrt{34}\pi$

e) $40\sqrt{34}$

Question 10

You did not answer the question.

Find the surface of revolution if the curve $x(t) = 5\,\cos(t),\ y(t) = 5\,\sin(t),$ for $t \in \left[0, \frac{\pi}{2}\right]$ is revolved around the x-axis.

$$\int_0^{\pi/2} 2\pi(5\sin t)\sqrt{25\sin^2 t + 25\cos^2 t}\,dt$$

$$\int_0^{\pi/2} 50\pi\sin t\,dt = -50\pi\cos t\Big|_0^{\pi/2}$$

$$0 - (-50\pi) = 50\pi$$

a) 5π

b) 25π

c) 100π

d) 50π

e) $\frac{5\pi}{2}$