1. Given $a_n = \left(\frac{(n+2)^{25}}{2^{n+3}}\right)\left(\frac{2^n}{(n+5)^{25}}\right)$, find $\lim\limits_{n\to\infty} a_n$. $\{a_n\}$

$$\lim_{n\to\infty} \frac{(n^{25}+\ldots\ldots)\cancel{2^n}}{(n^{25}+\ldots\ldots)\cancel{2^{n+3}}\,\underline{2^3}} = \frac{1}{8}$$

2. Determine if the sequence $\{a_n\}$ converges when $a_n = \frac{3n^2(2n-1)!}{(2n+1)!}$. If it converges, find the limit.

$$\lim_{n\to\infty} \frac{3n^2\cancel{(2n-1)!}}{(\underline{2n}+1)(\underline{2n})\cancel{(2n-1)!}} = \frac{3}{4}$$

3. Determine if the sequence $\{a_n\}$ converges when $a_n = \frac{12n^2}{3n+2} - \frac{4n^2+1}{n+2}$. If it converges, find the limit.

$\infty - \infty$

$$\lim_{n\to\infty} \frac{12n^2(n+2) - (4n^2+1)(3n+2)}{(3n+2)(n+2)}$$

$$= \lim_{n\to\infty} \frac{\cancel{12n^3} + \underline{24n^2} - \cancel{12n^3} - \underline{8n^2} - 3n - 2}{\underline{3n^2} + 8n + 4} = \frac{16}{3}$$

4. Determine if the sequence $\{a_n\}$ converges when $a_n = \frac{\ln(3n)}{\ln(5n)}$. If it converges, find the limit. $(\infty/\infty)$

$$\lim_{n\to\infty} \frac{\frac{3}{3n}}{\frac{5}{5n}} = \lim_{n\to\infty} 1 = 1$$

5. Determine if the sequence $\{a_n\}$ converges when $a_n = \frac{n^{4n}}{(n-2)^{4n}}$. If it converges, find the limit.

$$\lim_{n\to\infty}\left(\frac{n}{n-2}\right)^{4n} = \lim_{n\to\infty}\left(\frac{n-2}{n}\right)^{-4n} = \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{-4n}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x} \qquad = e^{(-2)(-4)} = \boxed{e^{8}}$$

6. Determine if the sequence $\{a_n\}$ converges when $a_n = (n^9)^{\frac{1}{5n}}$. If it converges, find the limit.

$$a_n = n^{\frac{9}{5n}} = \left(n^{1/n}\right)^{9/5}$$

as $n\to\infty$, $n^{1/n}\to 1$

$$1^{9/5} = \boxed{1}$$

7. Determine if the sequence $\{a_n\}$ converges when $a_n = \sqrt{n^3}-\sqrt{n^2}$. If it converges, find the limit. $\infty - \infty$

$$\lim_{n\to\infty}\frac{(\sqrt{n^3}-\sqrt{n^2})(\sqrt{n^3}+\sqrt{n^2})}{\sqrt{n^3}+\sqrt{n^2}} = \lim_{n\to\infty}\frac{n^3-n^2}{n^{3/2}+n}$$

$$\boxed{\text{diverges}}$$

8. Donny decides he is out of shape so he starts to run every day. His goal is to jog 4% more miles than the day before. How long will it take Donny to be able to run a total of 50 miles (for all days)? Assume he runs 1 mile on day 1.

$$1 + (1+.04(1)) + (1+.04(1)+.04[1+.04(1)]) + \ldots$$

$$a_1 + 1.04a_1 + 1.04(1.04a_1) + 1.04(1.04(1.04a_1))$$

$$1.04^0 a_1 + 1.04^1 a_1 + 1.04^2 a_1 + \cdots \quad \text{geom}$$

$$1 + 1.04 + 1.04^2 + 1.04^3 + \cdots = 50$$

$$\sum_{n=0}^{K} (1.04)^n = 1 + 1.04 + 1.04^2 + \cdots$$

find K so that sum is 50.

Finite geom. series $S_K = \dfrac{a_1(1-r^K)}{1-r}$

$$50 = \frac{1(1-1.04^K)}{1-1.04} = \frac{1-1.04^K}{-.04}$$

$$-2 = 1 - 1.04^K$$

$$1.04^K = 3$$

$$\ln 1.04^K = \ln 3$$

$$K(\ln 1.04) = \ln 3$$

$$K = \frac{\ln 3}{\ln 1.04} = 28.011 \rightarrow 29$$

$\boxed{\text{30 days}}$

9. Determine if the infinite series $\sum_{n=1}^{\infty} \ln\left(\frac{2n}{3n+1}\right)$ converges or diverges. If it converges, find the sum.

$$\ln\left(\frac{2n}{3n+1}\right) \to \ln 2/3 \neq 0$$

diverges by BDT

10. Determine if the infinite series $\sum_{n=1}^{\infty} (\cos^2\theta)^n$, $0 \le \theta < 2\pi$, converges or diverges. If it converges, find the sum.

$$\sum (r)^n \qquad |r| < 1$$

$$\frac{\cos^2\theta}{1-\cos^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$$

11. Find the rational representation of the repeating decimal $1.83838\overline{383}...$ using series.

$$\boxed{\frac{182}{99}}$$

$$1 + .83 + .0083 + .000083 + \cdots$$

$$1 + \left[\frac{83}{100^1} + \frac{83}{100^2} + \frac{83}{100^3} + \cdots\right]$$

$$1 + 83 \sum_{k=1}^{\infty}\left(\frac{1}{100}\right)^k = 1 + 83\left(\frac{1/100}{1-1/100}\right) = 1 + 83\left(\frac{1}{99}\right)$$

$$= 1 + \frac{83}{99}$$

12. Find the interval of all $x$ for which the series $\sum_{n=1}^{\infty} 2^n x^n$ converges.

$$\sum_{n=1}^{\infty} (2x)^n \qquad \text{Conv. when } |2x| < 1$$

$$-1 < 2x < 1$$

$$-1/2 < x < 1/2$$

$$\boxed{(-1/2, 1/2)}$$

13. If the nth partial sum of an infinite series, $\sum_{n=1}^{\infty} a_n$, is $S_n = \frac{2n}{n+1}$, find $a_n$.

$$\boxed{a_n = \frac{2}{n(n+1)}}$$

$$S_1 = \sum_{n=1}^{1} a_n = a_1 \rightarrow \frac{2}{2} = 1 \qquad a_1 = 1$$

$$S_2 = \sum_{n=1}^{2} a_n = a_1 + a_2 = \frac{4}{3}$$

$$1 + a_2 = 4/3$$

$$a_2 = 1/3$$

$$S_3 = S_2 + a_3$$

$$S_n = S_{n-1} + a_n$$

$$a_n = S_n - S_{n-1}$$

$$a_n = \frac{2n}{n+1} - \frac{2(n-1)}{n}$$

14. Determine if the infinite series $\sum_{n=1}^{\infty} \frac{3(n+1)^2}{n(n+4)}$ converges or diverges. If it converges, find the sum.

$$\frac{3(n+1)^2}{n(n+4)} \rightarrow 3 \neq 0$$

diverges by BDT

15. Determine if the infinite series $4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} \ldots$ converges or diverges. If it converges, find the sum.

$$\left(\frac{1}{2}\right)^{-2} \quad \left(\frac{1}{2}\right)^{-1} \quad \left(\frac{1}{2}\right)^{0} \quad \left(\frac{1}{2}\right)^{1}$$

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n-2} \quad \text{or} \quad \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n-2}$$

16. When applying the root test to an infinite series $\sum a_n$, we find the value of $\lim_{n\to\infty} |a_n|^{1/n} = \rho$.

Compute the value of $\rho$ for $\sum_{n=1}^{\infty} 2^{3n} \left(\frac{n-2}{n}\right)^{n^2}$.

$$\lim_{n\to\infty} \left[2^{3n}\left(\frac{n-2}{n}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty} 2^3 \left(\frac{n-2}{n}\right)^n$$

$$= \lim_{n\to\infty} 8\left(1 - \frac{2}{n}\right)^n = 8e^{-2}$$

17. When applying the root test to an infinite series $\sum a_n$, we find the value of $\lim_{n\to\infty} |a_n|^{1/n} = \rho$.

Compute the value of $\rho$ for $\sum_{n=1}^{\infty}\left(\frac{4\arctan n}{5}\right)^n$.

$$\lim_{n\to\infty}\left[\left(\frac{4\arctan n}{5}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{4}{5}\arctan(n)$$

$$= \frac{4}{5}\left(\frac{\pi}{2}\right) = \boxed{\frac{2\pi}{5}}$$

18. When applying the ratio test to an infinite series $\sum a_n$, we find the value of

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \rho$. Compute the value of $\rho$ for $\sum_{n=1}^{\infty}\frac{\sin(1/n)}{6n+11}$.

$\lim_{n\to 0}\frac{\sin n}{n} = 1 \quad // \quad \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$

$$\lim_{n\to\infty}\left(\frac{\sin\left(\frac{1}{n+1}\right)}{6n+17}\cdot\frac{6n+11}{\sin\left(\frac{1}{n}\right)}\right)$$

$$= \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}}\cdot\frac{\frac{1}{n}}{\sin\left(\frac{1}{n}\right)}\cdot\frac{6n+11}{6n+17}\cdot\frac{n}{n+1} = \boxed{1}$$

19. When applying the ratio test to an infinite series $\sum a_n$, we find the value of

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \rho$. Compute the value of $\rho$ for $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}\left(\frac{2}{7}\right)^n$.

$(n+1)\cancel{n!}\,(n+1)\cancel{n!}$

$$\lim_{n\to\infty}\frac{(n+1)!\,(n+1)!\,2^{\cancel{n+1}}}{(2n+2)!\,7^{\cancel{n+1}}}\cdot\frac{\cancel{(2n)!}\,\cancel{7^n}}{\cancel{n!}\,\cancel{n!}\,\cancel{2^n}}$$

$(2n+2)(2n+1)\cancel{(2n)!}$

$$= \lim_{n\to\infty}\frac{(n+1)(n+1)\,2}{(2n+2)(2n+1)\,7} = \frac{2}{28} = \boxed{\frac{1}{14}}$$

20. Use the integral test to determine if $\sum_{n=1}^{\infty}\frac{4}{n^2+1}$ converges or diverges.

$$\int_1^{\infty}\frac{4}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^{b}\frac{4}{x^2+1}\,dx = \lim_{b\to\infty} 4\tan^{-1}x \Big|_1^b$$

$$= \lim_{b\to\infty}\left(4\tan^{-1}(b) - 4\tan^{-1}(1)\right) = 4(\pi/2) - 4(\pi/4) = 2\pi - \pi$$

$$= \pi$$

21. If the improper integral $\int_1^{\infty}\frac{1}{x^p}dx$ converges, which of the following are always true?

a. $\boxed{\sum\frac{1}{n^p}\text{ converges}}$ ✓

b. $\sum\frac{1}{n^{p-1}}$ converges ÷

~~c.~~ $\sum\frac{1}{n^{p+1}}$ diverges

~~d.~~ $\sum\frac{1}{n^p}$ diverges

e. $\sum\frac{1}{n^{p-1}}$ diverges ÷

f. $\boxed{\sum\frac{1}{n^{p+1}}\text{ converges}}$ ✓

22. Determine if the following series converge or diverge:

Conv. a. $\sum_{n=0}^{\infty}\frac{4^n}{(n+2)^n}$ Root: $\lim_{n\to\infty}\frac{4}{n+2} = 0 < 1$

Conv. b. $\sum_{n=1}^{\infty}\left(\frac{3n}{4n+1}\right)^n\left(\frac{5}{4}\right)^n$ Root: $\lim_{n\to\infty}\left(\frac{3n}{4n+1}\right)\left(\frac{5}{4}\right) = \frac{15}{16} < 1$

Div. c. $\sum_{n=1}^{\infty}n!\left(\frac{3}{n}\right)^n$ Ratio: $\lim_{n\to\infty}(n+1)!\left(\frac{3}{n+1}\right)^{n+1}\cdot\frac{1}{n!}\left(\frac{n}{3}\right)^n$

$$\lim_{n\to\infty}\frac{\cancel{(n+1)}\cancel{n!}\,3^{\cancel{n+1}}\,n^n}{(n+1)^{\cancel{n+1}}\,\cancel{n!}\,\cancel{3^n}}$$

Conv. d. $\sum_{n=1}^{\infty}\frac{n^6+1}{n+2}\left(\frac{2}{9}\right)^n$

another page

$$\lim_{n\to\infty}3\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}3\left(\frac{n+1}{n}\right)^{-n}$$

$= \lim_{n\to\infty} 3\left(1+\frac{1}{n}\right)^{-n} = 3\cdot e^{-1} = \frac{3}{e} > 1$



Conv. e. $\sum_{n=1}^{\infty}\left(\frac{3n+7}{n^2+9}\right)^n$ Root: $\lim_{n\to\infty} \frac{3n+7}{n^2+9} = 0 < 1$

Div. f. $\sum_{n=1}^{\infty}\frac{\sqrt{n}-1}{\sqrt{n}+6}\left(\frac{3}{2}\right)^n$ another page

Div. g. $\sum_{n=1}^{\infty}\frac{(2n)^n}{n!}$ Ratio: $\lim_{n\to\infty} \frac{(2n+2)^{n+1}}{(n+1)\cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{(2n)^n}$

$= \lim_{n\to\infty} \frac{2^{\cancel{n+1}}(n+1)^{\cancel{n+1}}}{\cancel{(n+1)}\,\cancel{2^n}\cdot n^n}$

$= \lim_{n\to\infty} 2\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right)^n$

$= 2\cdot e > 1$

Conv. Root h. $\sum_{n=1}^{\infty}\frac{3^n}{(n+3)^n}$

Conv. i. $\sum_{k=1}^{\infty}\frac{2^k k^2}{k!}$

Div. BDT j. $\sum_{k=1}^{\infty}\frac{1}{(\sqrt[k]{3}-1)^k}$ $\left(\frac{1}{\sqrt[k]{3}}\right)^k = \frac{1}{3} \neq 0$

Conv. k. $\sum_{k=2}^{\infty}\frac{5}{k(\ln k)^5}$ $\int_2^{\infty} \frac{5}{x(\ln x)^5}\,dx = \lim_{b\to\infty}\left[\frac{-5}{4(\ln x)^4}\right]_2^b = 0 + \frac{5}{4(\ln 2)^4}$

$u = \ln x$, $du = \frac{1}{x}dx$ $\int \frac{1}{u^5}\,du = \frac{-1}{4u^4}$

23. Determine if the following series converge absolutely, converge conditionally or diverge:

ABS. CONV. a. $\sum_{n=0}^{\infty}\frac{(-3)^n}{(2n)!} = \sum \frac{(-1)^n 3^n}{(2n)!}$ alternates & $\frac{3^n}{(2n)!} \to 0 \Rightarrow$ converges

look at $\sum \left|\frac{(-1)^n 3^n}{(2n)!}\right| = \sum \frac{3^n}{(2n)!}$ use Ratio test

$\lim_{n\to\infty} \frac{3^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{3^n}$

ABS. CONV. b. $\sum_{n=1}^{\infty}\frac{n(-2)^n}{4^{n-1}}$

$= \lim_{n\to\infty} \frac{\cancel{3^n}\cdot 3\cdot \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{3^n}}$

COND. CONV. c. $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+3)\ln n}$ conv. by AST

$= \lim_{n\to\infty} \frac{3}{(2n+2)(2n+1)} = 0 < 1$

$\sum \frac{1}{(n+3)\ln n}$ — $\sum \frac{1}{n \ln n}$

doesn't conv. absolutely

$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_2^b$ diverges

DIV. (lim. comp) d. $\sum_{n=1}^{\infty} \frac{4}{8n+5}$ not alternating so either conv. absolutely or diverges $\sim \sum \frac{1}{n}$

Div. (BDT) e. $\sum_{n=1}^{\infty} \frac{4(-n)^n}{(n+5)^n} = \sum \frac{4(-1)^n n^n}{(n+5)^n}$ alt. but $4\left(\frac{n}{n+5}\right)^n \to 4 \cdot e^{-5} \neq 0$

Div. (geom or BDT) f. $\sum_{n=1}^{\infty}\left(-\frac{6}{5}\right)^n$

cond. conv. g. $\sum_{n=1}^{\infty}(-1)^n \frac{9}{3n\ln(n)+1}$ like part c

ABS. conv. h. $\sum_{n=1}^{\infty}\underbrace{\cos(\pi n)}_{\text{alt}}\left(\frac{2n}{3n+7}\right)^n$

does $a_n \to 0$

note: $\frac{2}{3+7/n}$ $n \geq 1$ $\frac{2}{3+7/n} \leq \frac{2}{3}$

$\boxed{(2/3)^n \to 0}$

abs? $\sum \left(\frac{2n}{3n+7}\right)^n$ Root: $\lim_{n\to\infty}\left(\frac{2n}{3n+7}\right) = \frac{2}{3} < 1$

Cond. Conv. i. $\sum_{k=1}^{\infty}(-1)^k \sin\left(\frac{1}{k}\right)$ alt $\sin 0 = 0$ $\sum \sin(1/k)$ comp. to $\sum \frac{1}{k}$

ABS. Conv. j. $\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\sin^2(k)}{4^k}$ $\sum \frac{\sin^2(k)}{4^k} < \sum \frac{1}{4^k}$ $\lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$ $y = 1/k$ $y \to 0$ $\lim_{y\to 0}\frac{\sin y}{y} = 1$

Cond. Conv k. $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}e^{1/k}}{4k}$ alt. + $\frac{e^{1/k}}{4k} \to 0$.

$\sum \frac{e^{1/k}}{4k} > \sum \frac{1}{4k} \sim \sum \frac{1}{k}$.

24. Which of the following statements are true:

F a. If $0 \leq a_n \leq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

F b. If $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges.

F c. The ratio test can be used to show that $\sum \frac{1}{n^3}$ converges.

$\lim_{n\to\infty} \frac{1}{(n+1)^3}\cdot\frac{n^3}{1} = 1$

25. Determine the radius and interval of convergence for the following:

$R = \infty$

$(-\infty, \infty)$

a. $\sum_{n=1}^{\infty} \frac{x^n}{(n+4)!}$

$0$

$\sum \frac{|x|^n}{(n+4)!}$ $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+5)!} \cdot \frac{(n+4)!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+5} = 0$

$R = 1$

$(5, 7)$

b. $\sum_{n=1}^{\infty} n^3 (x-6)^n$

$5 \quad 6 \quad 7$

$\sum n^3 |x-6|^n$ Root: $\lim_{n\to\infty} n^{3/n} |x-6| = |x-6| < 1$

$x = 5$: $\sum n^3 (-1)^n$ div.

$x = 7$: $\sum n^3$ div.

$R = 1$

$(2, 4)$

c. $\sum_{n=1}^{\infty} \sqrt{n}(x-3)^n$

$2 \quad 3 \quad 4$

$\sum \sqrt{n} |x-3|^n$ Root: $\lim_{n\to\infty} (n^{1/2})^{1/n} |x-3| = |x-3| < 1$

$x = 2$: $\sum \sqrt{n}(-1)^n$ div.

$x = 4$: $\sum \sqrt{n}$ div.

$R = 5$

$(-7, 3]$

d. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n 5^n}$

$\sum \frac{|x+2|^n}{n 5^n}$ Root: $\lim_{n\to\infty} \frac{|x+2|}{n^{1/n}\, 5} = \frac{|x+2|}{5} < 1$

$|x+2| < 5$

$-7 \quad -2 \quad 3$

$x = 3$: $\sum \frac{(-1)^n 5^n}{n \cdot 5^n} = \sum \frac{(-1)^n}{n}$ conv.

$x = -7$: $\sum \frac{(-1)^n (-5)^n}{n \cdot 5^n} = \sum \frac{(-1)^n \cdot (-1)^n \cdot \cancel{5^n}}{n \cdot \cancel{5^n}} = \sum \frac{1}{n}$ div.

$R = 0$

$\{1/3\}$

e. $\sum_{n=1}^{\infty} n!(3x-1)^n$

$\sum n! |3x-1|^n$

$1/3$

Ratio: $\lim_{n\to\infty} \frac{(n+1)!\, |3x-1|^{n+1}}{n!\, |3x-1|^n} = \lim_{n\to\infty} (n+1)|3x-1| = \infty$

$R = 0$

$R = 1$

$(-3/2, 1/2)$

f. $\sum_{n=1}^{\infty} \frac{(2x+1)^n n}{2^n}$

$\sum \frac{|2x+1|^n n}{2^n}$

Root: $\lim_{n\to\infty} \frac{|2x+1| n^{1/n}}{2} = \frac{|2x+1|}{2} < 1$

$|2x+1| < 2$

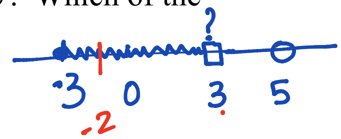
$|x + 1/2| < 1$

$-3/2 \quad -1/2 \quad 1/2$

$x = -3/2$: $\sum \frac{(-2)^n n}{2^n} = \sum (-1)^n n$ div.

$x = 1/2$: $\sum \frac{2^n n}{2^n} = \sum n$ div.

26. If the series $\sum_{n=0}^{\infty} c_n x^n$ converges when $x=-3$ and diverges when $x=5$. Which of the following series will converge without further restrictions on $c_n$?




a. $\sum_{n=0}^{\infty} c_n 7^n$ b. $\boxed{\sum_{n=0}^{\infty} c_n (-2)^n}$ ✓ c. $\sum_{n=0}^{\infty} c_n 3^n$

27. If the radius of convergence for the power series $\sum_{n=0}^{\infty} c_n x^n$ is 36, what is the radius of convergence for $\sum_{n=0}^{\infty} c_n x^{2n}$?

$|x| < 36$

$|x^2| < 36$

$\underline{\underline{6}}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

28. Find a power series representation for the given function centered at the origin:

a. $f(x) = \dfrac{1}{4+25x^2} = \dfrac{1}{4}\left(\dfrac{1}{1-\boxed{\frac{-25x^2}{4}}}\right) = \dfrac{1}{4}\sum_{n=0}^{\infty}\left(\dfrac{-25x^2}{4}\right)^n$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 25^n x^{2n}}{4^{n+1}}$$

b. $f(x) = \ln(3-x)$

$$f'(x) = \frac{-1}{3-x} = \frac{-1}{3}\left(\frac{1}{1-x/3}\right) = \frac{-1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)\,x^n}{3^{n+1}}$$

$$f(x) = \int f'(x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)\,x^n}{3^{n+1}}\,dx = \sum_{n=0}^{\infty}\frac{(-1)\,x^{n+1}}{(n+1)\,3^{n+1}} + \cancel{C}\ \ln 3$$

$$\underline{\underline{f(0) = \ln 3}} = 0 + \underline{\underline{C}}$$

c. $f(y) = \dfrac{y^3}{(1-y)^2}$

look at $\dfrac{1}{(1-y)^2}$

$$\frac{d}{dx}\left(\frac{1}{1-y}\right) = \frac{1}{(1-y)^2}$$

$$\frac{d}{dx}\sum_{n=0}^{\infty} y^n = \sum_{n=1}^{\infty} n\,y^{n-1} =$$

$$y^3\left(\frac{1}{(1-y)^2}\right) = y^3\sum_{n=1}^{\infty} n\cdot y^{n-1}$$

$$= \sum_{n=1}^{\infty} n\,y^{n+2}$$

d. $f(x)=\tan^{-1}\left(\frac{x}{4}\right)$

$f'(x)=\dfrac{1/4}{1+(x/4)^2}=\dfrac{1}{4}\left(\dfrac{1}{1-\left(-\frac{x^2}{16}\right)}\right)$

$=\dfrac{1}{4}\sum_{n=0}^{\infty}\left(\dfrac{-x^2}{4^2}\right)^n$

$f'(x)=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{4^{2n+1}}$

$f(x)=\int\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{4^{2n+1}}\,dx=\boxed{\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)4^{2n+1}}}+C$ $=0$

$f(0)=\tan^{-1}(0)=0$

e. $f(y)=\ln\left(\sqrt{\frac{1+2y}{1-2y}}\right)=\frac{1}{2}\left[\ln(1+2y)-\ln(1-2y)\right]$

$f'(y)=\dfrac{1}{2}\left(\dfrac{2}{1+2y}+\dfrac{2}{1-2y}\right)=\dfrac{2}{1-4y^2}=2\left(\dfrac{1}{1-4y^2}\right)=2\sum_{n=0}^{\infty}(4y^2)^n$

$f(y)=\int\sum_{n=0}^{\infty}2^{2n+1}y^{2n}\,dy=\boxed{\sum_{n=0}^{\infty}\dfrac{2^{2n+1}y^{2n+1}}{2n+1}}+C$ $=0$

$f(0)=\ln(1)=0$

f. $f(x)=\frac{2+x}{1-x}=-1+\dfrac{3}{1-x}$

$=-1+3\left(\dfrac{1}{1-x}\right)$

$=-1+3\sum_{n=0}^{\infty}x^n$

29. Find a function $f$ whose power series representation is $\sum_{n=2}^{\infty}n(n-1)x^{n+6}$ on $(-1,1)$.

$\sum_{n=0}^{\infty}x^n=\dfrac{1}{1-x}$

$\dfrac{d}{dx}\sum x^n=\sum nx^{n-1}$

$\dfrac{d}{dx}\sum nx^{n-1}=\sum n(n-1)x^{n-2}$

$\sum_{n=2}^{\infty}n(n-1)x^{n-2}\,x^8$

$=\dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)\right)\cdot x^8$

$\dfrac{2}{(1-x)^3}\cdot x^8=\boxed{\dfrac{2x^8}{(1-x)^3}}$

30. Evaluate the integral $f(x)=\int_0^x\frac{t}{1-t^4}\,dt$ as a power series.

$f'(x)=\dfrac{d}{dx}\int_0^x\dfrac{t}{1-t^4}\,dt=\dfrac{x}{1-x^4}=x\left(\dfrac{1}{1-x^4}\right)$

$=x\sum_{n=0}^{\infty}(x^4)^n$

$=\sum_{n=0}^{\infty}x^{4n+1}$

$f(x)=\int\sum_{n=0}^{\infty}x^{4n+1}\,dx=\boxed{\sum_{n=0}^{\infty}\dfrac{x^{4n+2}}{4n+2}}+C$ $=0$

22 d) $\boxed{\sum_{n=1}^{\infty} \frac{n^6+1}{n+2}\left(\frac{2}{9}\right)^n}$ Converges

$$\sum \frac{n^6}{n}\left(\frac{2}{9}\right)^n = \boxed{\sum n^5\left(\frac{2}{9}\right)^n} \text{ converges}$$

Root test: $\lim_{n\to\infty} n^{5/n}\left(\frac{2}{9}\right) = 2/9 < 1$

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Limit comparison of $\sum \frac{n^6+1}{n+2}\left(\frac{2}{9}\right)^n$

with $\sum n^5\left(\frac{2}{9}\right)^n$

$$\lim_{n\to\infty} \frac{\frac{(n^6+1)}{(n+2)}\cancel{\left(\frac{2}{9}\right)^n}}{n^5\cancel{(2/9)^n}} = \lim_{n\to\infty} \frac{n^6+1}{n^6+2n^5} = 1$$

positive

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f) $\sum_{n=1}^{\infty} \frac{\sqrt{n}-1}{\sqrt{n}+6}\left(\frac{3}{2}\right)^n$ diverges

$$\sum \frac{\sqrt{n}}{\sqrt{n}}\left(\frac{3}{2}\right)^n = \sum\left(\frac{3}{2}\right)^n \text{ diverges (geom)}$$

Limit comparison:

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}-1}{\sqrt{n}+6}\cancel{\left(\frac{3}{2}\right)^n}}{\cancel{\left(3/2\right)^n}} = 1 \quad \text{(positive)}$$

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i) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k^2}{k!}$ converges

Ratio: $\displaystyle\lim_{k\to\infty} \frac{2^{k+1}(k+1)^2}{(k+1)!} \cdot \frac{k!}{2^k \cdot k^2}$

$$= \lim_{k\to\infty} \frac{\cancel{2^k}\cdot 2\,(k+1)^{\cancel{2}}\ \cancel{k!}}{\cancel{(k+1)}\,\cancel{k!}\ \cancel{2^k}\cdot k^2}$$

$$\lim_{k\to\infty} \frac{2k+2}{k^2} = 0 < 1$$

23b) $\sum \frac{(-1)^n 2^n n}{4^{n-1}}$ alternates $\lim_{n\to\infty} \frac{2^n \cdot n}{4^{n-1}} = \lim_{n\to\infty} \frac{2^n \cdot n}{4^n \cdot 4^{-1}}$

$$= 4 \cdot \lim_{n\to\infty} \frac{2^n \cdot n}{4^n} \qquad \left(\frac{2}{4}\right)^n = \left(\frac{1}{2}\right)^n$$

$$= 4 \lim_{n\to\infty} \frac{n}{2^n} = \underline{0} \quad \Rightarrow \text{converges}$$

$$\sum \left| \frac{(-1)^n \cdot 2^n \cdot n}{4^{n-1}} \right| = 4 \underbrace{\sum \frac{2^n n}{4^n}}$$

Root test $\lim_{n\to\infty} \left(\frac{2^n n}{4^n} \cdot 4\right)^{1/n}$

$$= \lim_{n\to\infty} \frac{2 \cdot n^{1/n} \cdot 4^{1/n}}{4} = \underline{\frac{1}{2}} < \underline{1}$$

absolutely convergent