Math 1432

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Mondays 1-2pm,
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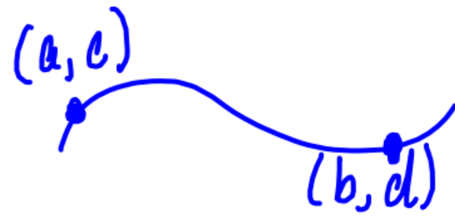
Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

Section 7.5:



$y = f(x)$

$x = g(y)$

<u>Arc Length Formula:</u>

$L = \int_C ds$

where $\quad ds = \sqrt{1 + (f'(x))^2}\, dx \quad$ for the curve C traced by $y = f(x), a \le x \le b$

or $\quad ds = \sqrt{1 + (g'(y))^2}\, dy \quad$ for the curve C traced by $x = g(y), c \le y \le d$

Example: Find the length of the graph of

$f(x) = \frac{1}{3}x^{3/2} - \sqrt{x}$ from $x = 1$ to $x = 9$

$$(f'(x))^2 = \left(\frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}\right)^2$$

$$\frac{1}{4}x - \frac{1}{2} + \frac{1}{4x}$$

$$L = \int_1^9 \sqrt{1 + \left(\frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$+\,2/4$$

$$= \int_1^9 \sqrt{\frac{4}{4} + \frac{x}{4} - \frac{2}{4} + \frac{1}{4x}}\, dx$$

$$= \int_1^9 \sqrt{\frac{x^2 + 2x + 1}{4x}}\, dx$$

$$= \int_1^9 \sqrt{\frac{(x+1)^2}{4x}}\, dx$$

$$\int_1^9 \frac{(x+1)}{2\sqrt{x}}\,dx = \int_1^9 \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right)dx$$

$$= \frac{1}{3}x^{3/2} + x^{1/2}\Big|_1^9$$

$$= \left(\frac{1}{3}(9)^{3/2} + 9^{1/2}\right) - \left(\frac{1}{3} + 1\right)$$

$$= 9 + 3 - 1/3 - 1$$

$$= 10\tfrac{2}{3} \text{ or } 32/3$$

Find the length of the graph of $f(x)=\underbrace{\frac{1}{4}x^2-\frac{1}{2}\ln x}$ from $x=1$ to $x=2$

$$\left(\frac{x}{2}-\frac{1}{2x}\right)\left(\frac{x}{2}-\frac{1}{2x}\right)$$

$$f'(x)=\frac{x}{2}-\frac{1}{2x}$$

$$-\frac{x}{4x}-\frac{x}{4x}$$

$$L=\int_1^2\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,dx=\int_1^2\sqrt{\frac{4}{4}+\frac{x^2}{4}-\frac{2}{4}+\frac{1}{4x^2}}\,dx$$

$$+\frac{2}{4}$$

$$=\int_1^2\sqrt{\frac{x^4+2x^2+1}{4x^2}}\,dx=\int_1^2\sqrt{\frac{(x^2+1)^2}{4x^2}}\,dx$$

$$=\int_1^2\frac{x^2+1}{2x}\,dx=\int_1^2\frac{x}{2}+\frac{1}{2x}\,dx=\left[\frac{x^2}{4}+\frac{1}{2}\ln|x|\right]_1^2$$

$$\frac{1}{2}\cdot\frac{1}{x}$$

$$=\left(1+\frac{1}{2}\ln 2\right)-\left(1/4+\frac{1}{2}\ln 1\right)=\boxed{\frac{3}{4}+\frac{1}{2}\ln 2}$$

$$\frac{\cos^2 x + \sin^2 x = 1}{\cos^2 x}$$

$$\underbrace{1 + \tan^2 x} = \sec^2 x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 + \sinh^2 x = \cosh^2 x$$

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$$L = \int_0^{\pi/4} \sqrt{1 + (\tan x)^2}\, dx$$
$$= \int_0^{\pi/4} \sec x\, dx$$

1. Find the length of the curve $f(x) = \ln|\sec x|$ for $0 \le x \le \pi/4$

(hint: $\int \sec x\, dx = \ln|\sec x + \tan x| + C$)

$$f' = \frac{\cancel{\sec x}\tan x}{\cancel{\sec x}}$$

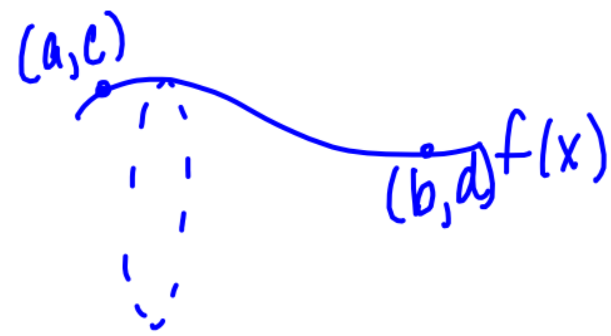
a. $\ln(\sqrt{2}+1)$

b. $\ln(\sqrt{2})$

$$\sec \pi/4 = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}$$

c. $\ln(\sqrt{2}+1) - \ln(2)$

d. none of these



Surface Area

$$A(S)=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx. \leftarrow$$

$$A(S)=\int_c^d 2\pi F(y)\sqrt{1+[F'(y)]^2}\,dy.$$

Examples:

Find the area of the surface generated by revolving the curve $f(x)=4/3\ x$ with $0\le x\le 3$ about the x-axis.

$1+\frac{16}{9}=\frac{25}{9}$

$f'(x)=4/3$

$$A(S)=\int_0^3 2\pi\left(\frac{4}{3}x\right)\sqrt{1+(4/3)^2}\,dx=\int_0^3 \frac{8}{3}\pi x\sqrt{\frac{25}{9}}\,dx$$

$$\frac{40}{9}\pi\left.\frac{x^2}{2}\right|_0^3=\frac{40}{9}\pi\left(\frac{9}{2}-0\right)=\boxed{20\pi}$$

Find the area of the surface generated by revolving the curve $f(x)=2x^3$ with $0 \le x \le 2$

$f'(x) = 6x^2$

$[f'(x)]^2 = 36x^4$

$u = \overbrace{1+36x^4}$

$du = 144x^3 dx$

$A(S) = \int_0^2 2\pi (2x^3)\sqrt{1+36x^4}\, dx = \frac{4\pi}{144}\int_0^2 144x^3\sqrt{1+36x^4}\,dx$

$\frac{\pi}{36}\int_1^{577} \sqrt{u}\, du = \underbrace{\frac{\pi}{36}\cdot\frac{2}{3}}\, u^{3/2} \Big|_1^{577}$

$\frac{\pi}{54}\left(577^{3/2} - 1\right)$

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2. Which of the following integrals will find the surface area of the curve $f(x)=\sqrt{x}$ for $0 \le x \le 2$ revolved about the x axis?

a. $\int_0^2 \pi(\sqrt{x})^2\,dx$

b. $\int_0^2 2\pi\sqrt{x}\sqrt{1+\frac{1}{2x}}\,dx$

c. $\int_0^2 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx$

d. none of these

Centroids and Centers of Mass

Question: Where is the center of mass if the rectangle is made from homogeneous material?

Answer: Dead Center.

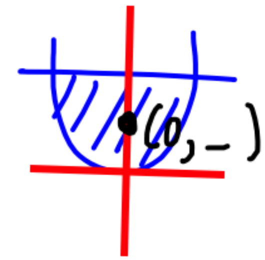
Notice that this is also the balancing point.

The center of mass for nonhomogeneous material is always the balancing point.

The centroid is the balancing point when the region is treated as homogeneous.

The centroid $(\bar{x}, \bar{y})$ of a region R can be obtained by:

$$\bar{x} = \frac{\int_a^b x[f(x) - g(x)]\, dx}{A} \text{ and } \bar{y} = \frac{\int_a^b \frac{1}{2}\left[\left([f(x)]^2 - [g(x)]^2\right)\right] dx}{A}$$



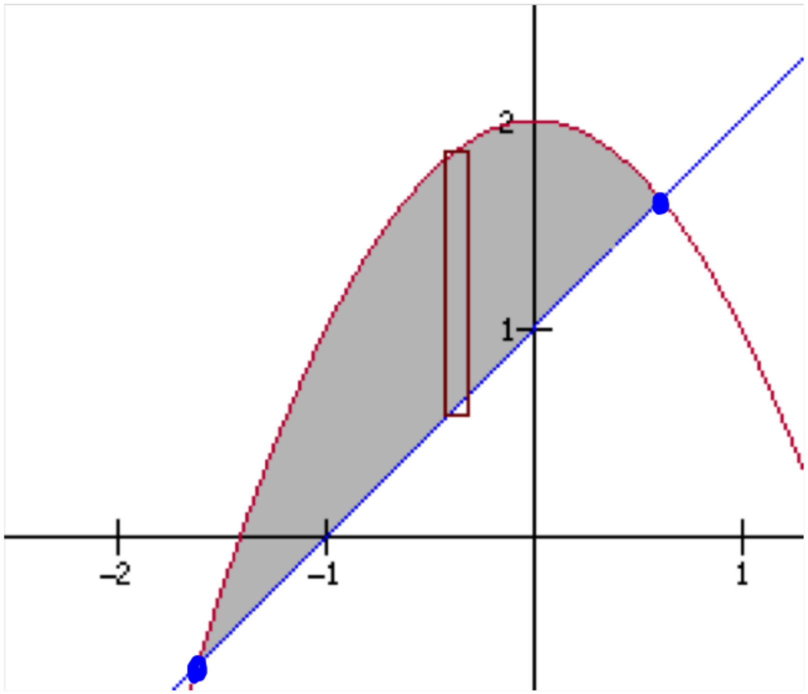
Where A is the area of the region.

Find the centroid of the region bounded by y=x+1 and y=2-x^2

$x+1=2-x^2$

$x^2+x-1=0$

$x=\frac{-1\pm\sqrt{5}}{2}$



$$A=\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}}(2-x^2)-(x+1)\,dx$$

$$\bar{x}=\frac{\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} x\left[(2-x^2)-(x+1)\right]dx}{A}$$

$$\bar{y}=\frac{\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} \frac{1}{2}\left[(2-x^2)^2-(x+1)^2\right]dx}{A}$$

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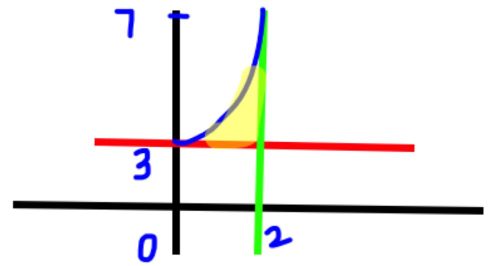


$$A = \int_0^2 x^2+3-3\,dx = \frac{8}{3}$$

3. Determine the centroid of the region: $y=x^2+3, y=3$, and $x=2$

a. (21/5, 3/2)

b. (3/4, 21/5)

c. (3/2, 21/5)

d. (3/4, 21/10)

$$\bar{x} = \frac{\int_0^2 x\left(x^2+3-3\right)dx}{8/3} = \frac{\int_0^2 x^3\,dx}{8/3} = \frac{3}{2}$$

$$\bar{y} = \frac{\int_0^2 \frac{1}{2}\left(\left(x^2+3\right)^2-3^2\right)dx}{8/3} = \frac{\frac{1}{2}\int_0^2 x^4+6x^2\,dx}{8/3} = \frac{21}{5}$$

Theorem 7.5.1: Pappus's Theorem on Volumes:
Suppose a solid is created by revolving region Ω in the plane around any axis, such that Ω does not cross this axis. Then the volume of the solid is given by:

$$V = 2\pi \bar{R} A$$

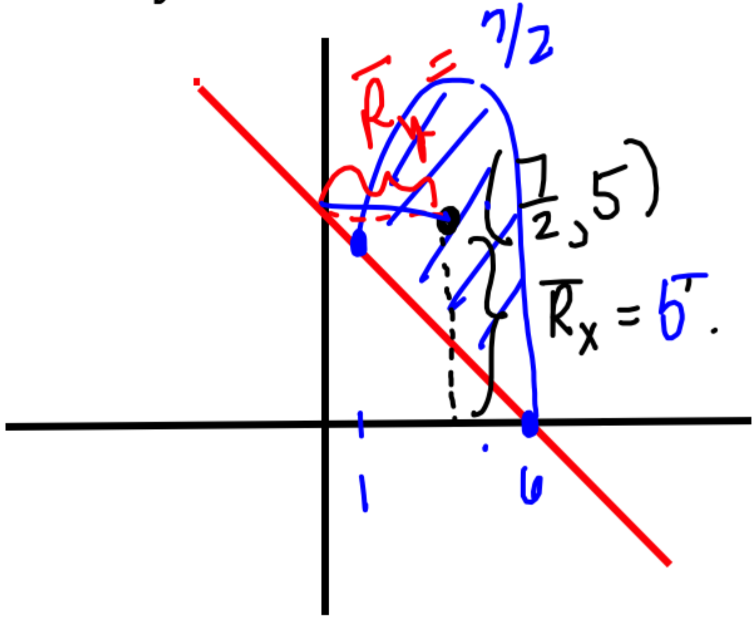
Where $\bar{R}$ is the distance from the centroid of Ω to the axis of revolution and A is the area of the region Ω.

Find the centroid of the region bounded by

$$y = 6x - x^2 \text{ and } x + y = 6.$$

Then find the volume of the region when revolved about the x-axis and then the y-axis.

$$6x - x^2 = 6 - x$$
$$0 = x^2 - 7x + 6$$
$$0 = (x-6)(x-1)$$



$$A = \int_1^6 \left[6x - x^2 - (6 - x)\right] dx = \frac{125}{6}$$

$$\bar{x} = \frac{\int_1^6 x\left[6x - x^2 - (6-x)\right] dx}{125/6} = \frac{7}{2}$$

$$\bar{y} = \frac{\int_1^6 \frac{1}{2}\left[(6x - x^2)^2 - (6-x)^2\right] dx}{125/6} = 5$$

$$V_{x\,axis} = 2\pi(5)\left(\frac{125}{6}\right) = \frac{625\pi}{3}$$

$$V_{y\,axis} = 2\pi\left(\frac{7}{2}\right)\left(\frac{125}{6}\right) = 875$$

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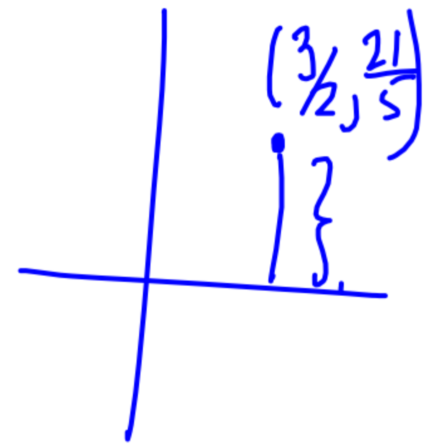
4. Use Pappus' thm to find the volume of the solid formed when the region $y=x^2+3, y=3$, and $x=2$ is rotated around the x-axis.

5. Use Pappus' thm to find the volume of the solid formed when the region $y=x^2+3, y=3$, and $x=2$ is rotated around the y-axis.

Answer choices for both:

$C\ \left(\frac{3}{2}, \frac{21}{5}\right)$



a. 8π

b. $\frac{8}{3}\pi$

c. $\frac{336}{5}\pi$

d. $\frac{112}{5}\pi$

$V_x = 2\pi\left(\frac{21}{5}\right)\frac{8}{3}$

$V_y = 2\pi\left(\frac{3}{2}\right)\frac{8}{3}$