Math 1432

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Office Hours:

Mondays 1-2pm,
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(also available by appointment)

Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

Popper06

if $\lim_{x \to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then $= \lim_{x \to a} \frac{f'(x)}{g'(x)}$

1. Compute $\lim_{x \to 0} \frac{x - \sin x}{x - \tan x}$. $\frac{0}{0}$

a. 1/2

b. –1/2

c. 1

d. 0

e. DNE

$$\lim_{x \to 0} \frac{1 - \cos x}{1 - \sec^2 x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\sin x}{-2\sec x \cdot \sec x \, \cancel{\tan x}} \quad \frac{\sin x}{\cos x}$$

2. Compute $\lim_{x\to 0} \dfrac{e^{x} - e^{-2x}}{2\sin x}$. $\dfrac{e^0 - e^0}{2\sin 0} = \dfrac{0}{0}$

Use L.R.

a. 1

b. 2

c. 3

d. 3/2

e. DNE

3. Compute $\lim_{x \to \infty} \left(\arctan(x) \right)$.

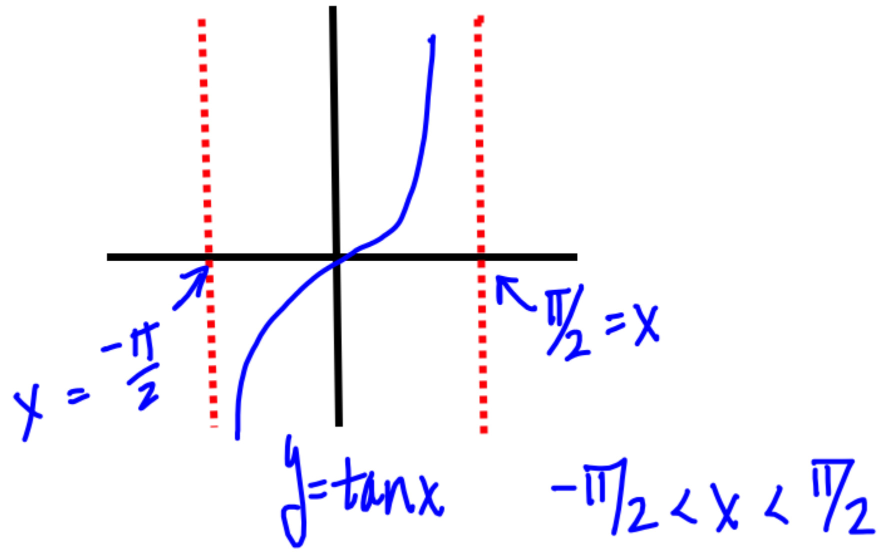
a. $\dfrac{\pi}{4}$

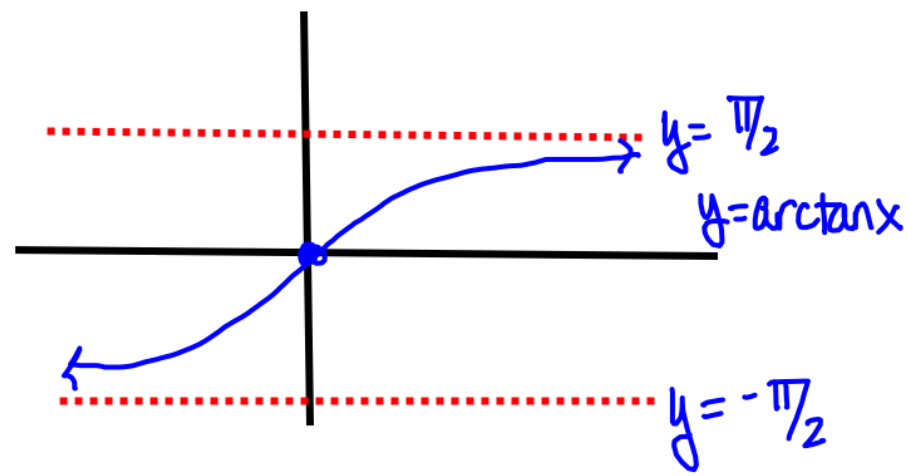
b. 0

c. $\dfrac{\pi}{2}$

d. 1

e. DNE





Improper Integrals

The definition of the definite integral $\int_a^b f(x)dx$ requires that [a, b] be finite and that f(x) be bounded on [a, b].

Also, the Fundamental Theorem of Calculus requires that f be continuous on [a, b].

If one or both of the limits of integration are infinite or if f has a finite number of infinite discontinuities on [a, b], then the integral is called an improper integral.

Types of improper integrals:

A. (one or both bounds are infinite)

$\int_{1}^{\infty} \frac{dx}{x}$, $\int_{-\infty}^{1} \frac{3dx}{x^4+5}$ and $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ are improper because one or both bounds are infinite.

B. (infinite discontinuity at a boundary)

$\int_{1}^{5} \frac{dx}{\sqrt{x-1}}$ is improper because $f(x) = \frac{1}{\sqrt{x-1}}$ has an infinite discontinuity at $x = 1$.

C. (infinite discontinuity in the interior)

$\int_{-2}^{2} \frac{dx}{(x+1)^2}$ is improper because $f(x) = \frac{1}{(x+1)^2}$ has an infinite discontinuity at $x = -1$, and -1 is between -2 and 2.

For the first type of improper integrals:

1) If f is continuous on $[a,\infty)$, then

$$\int_a^{\infty} f(x)\,dx = \boxed{\lim_{b\to\infty}} \int_a^{b} f(x)\,dx.$$

2) If f is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \underline{\lim_{a\to-\infty}} \int_a^{b} f(x)\,dx.$$

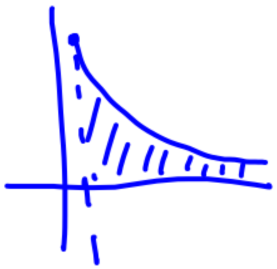
3) If f is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx.$$

If the limit exists, then the improper integral is said to converge.
Otherwise, it diverges.

Examples for the first type of improper integral.

1. $\int_1^{\infty} \frac{dx}{x}$ $= \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \left[\ln|x| \right]_1^b$



$= \lim_{b\to\infty} \left[\ln b - \ln 1 \right] = \lim_{b\to\infty} (\ln b) \to \infty$

so the integral diverges

2. $\int_2^{\infty} e^{-x}\,dx$ $= \lim_{b\to\infty} \int_2^b e^{-x}\,dx$

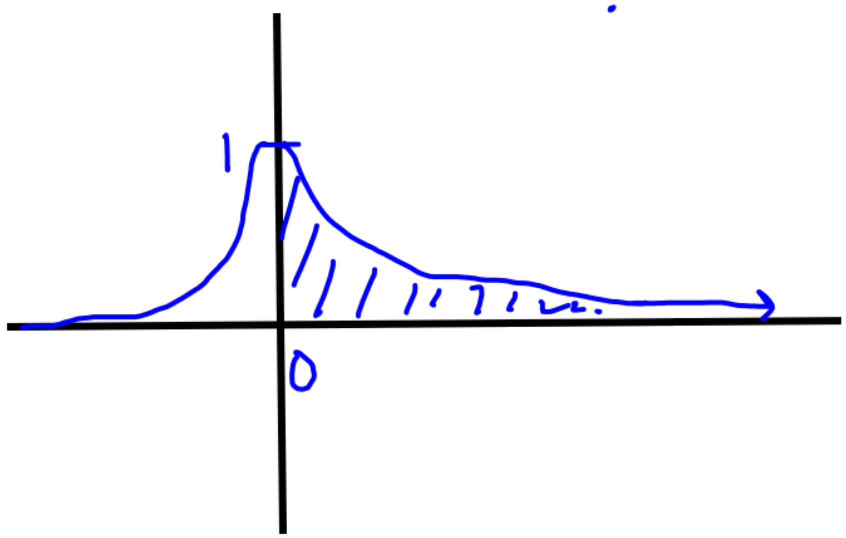
$= \lim_{b\to\infty} \left[-e^{-x} \right]_2^b = \lim_{b\to\infty} \left[-e^{-b} - -e^{-2} \right]$

$= \lim_{b\to\infty} \left[\overbrace{-\frac{1}{e^b}}^{0} + \frac{1}{e^2} \right] = 0 + \frac{1}{e^2} = \boxed{\frac{1}{e^2}}$

3. $\int_0^\infty \frac{1}{x^2+1}dx = \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}dx = \lim_{b\to\infty} \arctan x \Big|_0^b$

$= \lim_{b\to\infty} (\arctan b - \arctan 0)$

$= \lim_{b\to\infty} \arctan b = \boxed{\frac{\pi}{2}}$



4. $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ $= \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+(e^x)^2}\,dx$

$$\lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^{2x}}\,dx + \lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^{2x}}\,dx$$

$$\left.\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}\right\} \int \frac{1}{1+u^2}\,du = \arctan u$$

$$\lim_{a\to-\infty} \left[\arctan e^x\right]_a^0 + \lim_{b\to\infty} \left[\arctan e^x\right]_0^b$$

$$\lim_{a\to-\infty} \Big[\underbrace{\arctan(1)}_{\pi/4} - \arctan(\underbrace{e^a}_{e^{-big} = \frac{1}{e^{big}}}) \xrightarrow{} 0\Big] + \lim_{b\to\infty} \Big[\arctan e^b - \underbrace{\arctan e^0}_{\pi/4}\Big]$$

$$\pi/4 - 0 + \pi/2 - \pi/4 = \boxed{\pi/2}$$

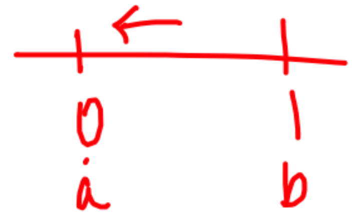
The second and third type of improper integral:

1. If f is continuous on $\left[a,b \right)$ but has an infinite discontinuity at b, then
$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx.$$

2. If f is continuous on $\left(a,b \right]$ but has an infinite discontinuity at a, then
$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx.$$

3. If f is continuous on $[a, b]$ except for some c in (a, b) at which f has an infinite discontinuity, then
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$
provided **both** integrals on the right converge. If either integral on the right diverges, we say that the integral on the left diverges.

Examples for the second type of improper integral.



1. $\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{a\to 0+} \int_a^1 x^{-1/3}\,dx$

$$= \lim_{a\to 0+} \frac{3}{2} x^{2/3} \Big|_a^1 = \lim_{a\to 0+} \left[\frac{3}{2} - \frac{3}{2} a^{2/3} \right]$$

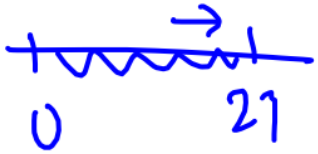
$$= \frac{3}{2}$$

2. $\int_0^2 \frac{dx}{x^3}$ $= \lim_{a \to 0^+} \int_a^2 x^{-3}\,dx = \lim_{a \to 0^+} \left. \frac{x^{-2}}{-2} \right|_a^2$

$= \lim_{a \to 0^+} \left[\frac{-1}{8} + \frac{+1}{2a^2} \right] =$ divergesDNE

@ $a=0$

3. $\displaystyle\int_0^{27} \frac{dx}{\sqrt[3]{27-x}} = \lim_{b \to 27^-} \int_0^{b} (27-x)^{-1/3}\,dx$



$$= \lim_{b \to 27^-} \left[-\frac{3}{2}(27-x)^{2/3} \right]_0^b$$

$$= \lim_{b \to 27^-} \left[-\frac{3}{2}(27-x)^{2/3} + \frac{3}{2}(27)^{2/3} \right]$$

$$0 + \frac{27}{2}$$

$$= \boxed{\frac{27}{2}}$$

4. $\displaystyle\int_1^4 \frac{dx}{x-2} = \int_1^2 \frac{dx}{x-2} + \int_2^4 \frac{dx}{x-2}$

$$= \lim_{b\to 2^-} \int_1^b \frac{dx}{x-2} + \lim_{a\to 2^+} \int_a^4 \frac{dx}{x-2}$$

$$= \lim_{b\to 2^-} \Big[\ln|x-2|\Big]_1^b + \lim_{a\to 2^+} \Big[\ln|x-2|\Big]_a^4$$

$$= \lim_{b\to 2^-} \Big[\ln|b-2| - \ln(1)\Big] + \lim_{a\to 2^+} \Big[\ln(2) - \ln|a-2|\Big]$$

ln 0
undef.

diverges

Important examples:

$$\int_1^\infty \frac{dx}{x^p} \qquad p = 1$$

$$\int_1^\infty \frac{dx}{x^p} \qquad p > 1$$

$$\int_1^\infty \frac{dx}{x^p} \qquad 0 < p < 1$$

$$\int_1^\infty \frac{dx}{x^p}$$

Diverges for $p \leq 1$
Converges for $p > 1$

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Which of the following are improper integrals?

4. $\int_{2}^{3} \frac{1}{x^{2/3}}\,dx$ **a) yes** **b) no**

5. $\int_{-1}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **a) yes** **b) no**

6. $\int_{1}^{3} \frac{1}{(x-2)^2}\,dx$ **a) yes** **b) no**

7. $\int_{0}^{\infty} e^{-2x}\,dx$ **a) yes** **b) no**