Math 1432

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Office Hours:

Mondays 1-2pm,
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Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$$

$$\frac{a}{1-r} - a\frac{(1-r)}{(1-r)} = \frac{\cancel{a - a} + ar}{1-r}$$

If $a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} = \sum_{n=1}^{\infty} \frac{-1/2}{n+3} + \frac{1/2}{n+1}$$

or

$$\sum_{n=1}^{\infty} \frac{-1}{2n+6} + \frac{1}{2n+2}$$

$$\frac{1}{(n+3)(n+1)} = \frac{A}{n+3} + \frac{B}{n+1}$$

$$A(n+1) + B(n+3) = 1$$

$n=-1$: $\quad 0 + 2B = 1 \qquad B = 1/2$

$n=-3$: $\quad -2A + 0 = 1 \qquad A = -1/2$

$$= \left(\frac{-1}{\cancel{8}} + \frac{1}{4}\right) + \left(\frac{-1}{\cancel{10}} + \frac{1}{6}\right) + \left(\frac{-1}{12} + \frac{1}{\cancel{8}}\right) + \left(\frac{-1}{14} + \frac{1}{\cancel{10}}\right) + \cdots$$

$$= 1/4 + 1/6 = \frac{5}{12}$$

Popper 21

1. $$\left\{ \frac{2n^2}{n^2+6n} \right\}_{n=1}^{\infty}$$

a. converges

b. diverges

2. $$\sum_{n=1}^{\infty} \frac{2n^2}{n^2+6n}$$

a. converges

b. diverges

3. $\left\{\left(1-\dfrac{1}{n}\right)^n\right\}_{n=1}^{\infty} \to e^{-1}$

$\left(1+\dfrac{x}{n}\right)^n \to e^x$

as $n \to \infty$

a. converges

b. diverges

4. $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$.

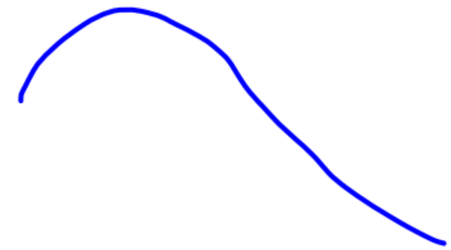
a. converges

b. diverges

Section 9.4

The Integral Test; Comparison Tests

Integral Test ("hardest" test – be careful!):



If f is **positive**, **continuous**, and (*ultimately*) **decreasing** for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either **both** converge or both diverge.

Note: When we use the Integral Test it is not necessary to start the series or the integral at n = 1.

Also, it is not necessary that f be always decreasing. What is important is that f be *ultimately* decreasing.

That is, decreasing for x larger than some number N, since a finite number of terms doesn't affect the convergence or divergence of a series.

Examples: Determine whether the following series converge or diverge. Show that the series meets the requirements of the integral test before you use it.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

diverges

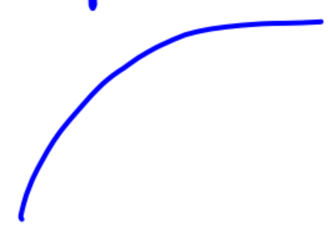
$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_1^{b} \frac{2x}{(x^2+1)}\,dx$$

$u = x^2+1$
$du = 2x\,dx$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(x^2+1)\Big]_1^b$$

$$= \lim_{b\to\infty} \Big[\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln(2)\Big]$$

$\to \infty$ (diverges)



2) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Converges

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \arctan(x)\Big|_1^b$$

$$= \lim_{b\to\infty} \left(\arctan(b) - \pi/4\right)$$

$$= \pi/2 - \pi/4 = \boxed{\pi/4}$$

Conv.

3) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

conv.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1$$

4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

diverges

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b x^{-1/2}\,dx$$

$$= \lim_{b\to\infty} 2x^{1/2}\Big|_1^b$$

$$= \lim_{b\to\infty} \left(2\sqrt{b} - 2\right) \to \infty$$

5) Use the integral test to determine the values of p for which

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges}$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx$$

$(p \neq 1)$

$$\lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b$$

$$= \lim_{b\to\infty} \underbrace{\frac{1}{b^{(p-1)}(-p+1)}}_{\to 0 \text{ if } p-1>0} - \frac{1}{-p+1}$$

$p > 1$

then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv.

$p \leq 1 \Rightarrow$ div.

when $p = 1$

$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}$ diverges ☆ — Harmonic Series

☆ $-p+1 = -(p-1)$

p-Series Test:

$\sum \frac{1}{n^2}$ or $\sum \frac{1}{\sqrt{n}}$

Geom
$\sum \left(\frac{2}{3}\right)^n$

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

is called a **p-series**, where p is a positive constant.

For p = 1, the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is called the **harmonic series**.

The **harmonic series diverges.**

The p-series **diverges** if $0 < p \leq 1$.

The p-series **converges** if $p > 1$.

Examples: Determine whether the following series converge or diverge.

1) $\sum_{n=1}^{\infty} \frac{1}{n}$. diverge

2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$. $p = 1/4$ diverges

3) $\sum_{n=1}^{\infty} \frac{1}{n^3\sqrt{n}}$ $= \sum \frac{1}{n^{7/2}}$ $p = 7/2$ converges

$\sum \frac{1}{n^2+1}$ $\sum \frac{1}{n^2}$

Basic Comparison Test:

If $a_n \geq 0$ and $b_n \geq 0$ and

1) If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

less than conv. will converge

2) If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

greater than divergent diverges

So.... $\sum a_n \quad \sum b_n$

Let $a_n \geq 0$ and $b_n \geq 0$,

If A diverges and B < A, what happens? ??

If A converges and B > A, what happens? ??

If A converges and B < A, what happens? B conv.

If A diverges and B > A, what happens? B diverges

$$\frac{1}{2} > \frac{1}{3} \qquad \frac{1}{n^2} > \frac{1}{n^2+1}$$

Examples: Determine whether the following series converge or diverge.

BCT

1) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ compare to $\sum \frac{1}{n^3}$ conv. p series ($p=3$)

converges

BCT

2) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ compare to $\sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$

geom conv.

$|1/3|<1$

converges

3) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}-3}$ $\sim$ $\sum \frac{1}{\sqrt{n}}$ div. pseries $p = 1/2$

diverges

$$\frac{1}{\sqrt{n}-3} > \frac{1}{\sqrt{n}}$$

Popper 21

5. Use the fact that $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}$ $= \frac{\pi^2}{6}$ to find $\sum_{n=3}^{\infty} \frac{1}{n^2}$ $= \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

$$1 + \frac{1}{4} + \underbrace{\frac{1}{9} + \frac{1}{16}}$$

a) $\dfrac{2\pi^2 - 15}{12}$

b) $\dfrac{\pi^2 - 6}{6}$

c) $\dfrac{\pi^2 - 12}{6}$

d) $\dfrac{\pi^2}{6}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{1}{n^2}$$

6. $\sum_{n=1}^{\infty} \frac{1}{n^7}$

a. converges

b. diverges

7. $\sum_{n=1}^{\infty} \frac{1}{n+3}$

a. converges

b. diverges