Math 1432

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Office Hours:

Mondays 1-2pm,
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(also available by appointment)

Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

Geometric Series Test. $\sum_{\square} (r)^n$ conv. when $|r|<1$

Basic Divergence Test. if $a_n \not\to 0$ then $\sum a_n$ diverges

p-Series Test. $\sum \frac{1}{n^p}$ conv. if $p>1$

Integral Test. if $\int_1^\infty f(x)\,dx$ conv./div. then $\sum f(n)$ conv./div.

Basic Comparison Test. less than conv. converges
greater than div. diverges

Limit Comparison Test. $\lim_{n\to\infty} a_n/b_n$ = finite positive $\Rightarrow$ $\sum a_n$ & $\sum b_n$ do same

Root Test $\lim_{n\to\infty} (a_n)^{1/n}$ <1 conv, >1 div.

Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ <1 conv., >1 div

(Root and Ratio Tests:) $=1 \to$ inconclusive

Alternating Series Test for Convergence: $\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad b_n > 0$

if Σa_n alternates + terms $\to 0 \Rightarrow$ converges

$\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges

but $\sum_{n=1}^{\infty} |a_n|$ diverges.

(Note: a non-alternating series can never converge conditionally)

Popper 26

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2}$ ← alternates

& $\frac{1}{2n^2} \to 0 \Rightarrow$ conv.

(a.) converges absolutely

b. converges conditionally

~~c. diverges~~

ck for abs: $\sum \left| \frac{(-1)^n}{2n^2} \right| = \sum \frac{1}{2n^2}$

2. $\sum_{n=1}^{\infty} \frac{2n+1}{5n^2+2n}$ $\sim$ $\sum \frac{1}{n}$

a. converges

b. diverges

3. $\sum_{n=1}^{\infty} \frac{3n+1}{5n^3+2n}$ $\sim$ $\sum \frac{1}{n^2}$

a. converges

b. diverges

4. $$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

a. converges

b. diverges

5. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ $\leftarrow$ alt

$+ \frac{1}{n} \to 0$

$\Rightarrow$ conv

a. converges absolutely ? $\sum \frac{1}{n}$

b. converges conditionally

~~c. diverges~~

6. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \arctan(n)}{n^2}$$ $< \pi/2$

$$\sum \frac{(-1)^{n+1}}{n^2}$$

a. converges absolutely

b. converges conditionally

$$\sum \frac{1}{n^2}$$

~~c.~~ diverges

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$ alt.

$\frac{n}{2^n} \to 0$ ✓

a. converges absolutely

b. converges conditionally

$\sum \frac{n}{2^n}$

~~c.~~ diverges

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{n^2+1}$ alt

$\frac{n}{n^2+1} \to 0$ ✓

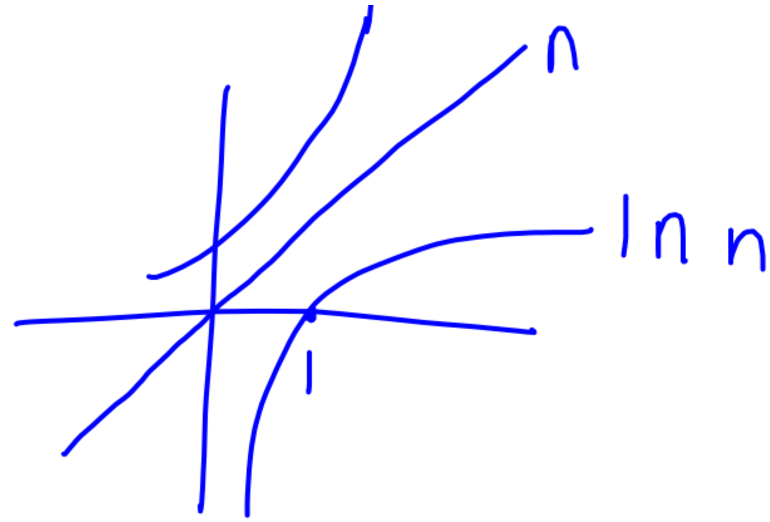
a. converges absolutely

b. converges conditionally

$\sum \frac{n}{n^2+1}$

~~c.~~ diverges

9. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $> \sum \frac{1}{n}$



a. converges

b. diverges

alt +

10. $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ $\frac{\ln n}{n} \to 0$

a. converges

b. diverges

Notes for series "growth":

Let $p(k)$ be a polynomial in k. (k to a power (const))

r^k for $r > 1$ grows much faster than $p(k)$ ex 2^k grows faster than k^5

$k!$ grows much faster than r^k, $p(k)$

k^k grows much faster than the others

Hence,

$$\sum \frac{p(k)}{r^k}, \quad \sum \frac{p(k)}{k!}, \quad \sum \frac{p(k)}{k^k}$$

$$\sum \frac{r^k}{k!}, \quad \sum \frac{r^k}{k^k}, \quad \sum \frac{k!}{k^k}$$

ALL converge rapidly.

Power Series:

Suppose that $f(x) = \dfrac{6}{1-x}$. $\dfrac{a_1}{1-r}$

$$\begin{array}{r}
6 + 6x \\
1 - x \enclose{longdiv}{6} \\
-\ 6 + 6x \\
\hline
6x \\
-\ 6x + 6x^2 \\
\hline
6x^2
\end{array}$$

If you divide 1 – x into 6, you get a "polynomial" that continues forever.

$$P(x) = 6 + 6x + 6x^2 + 6x^3 + 6x^4 + \ldots$$

$$6 \sum_{n=0}^{\infty} x^n$$

This result is a power series.

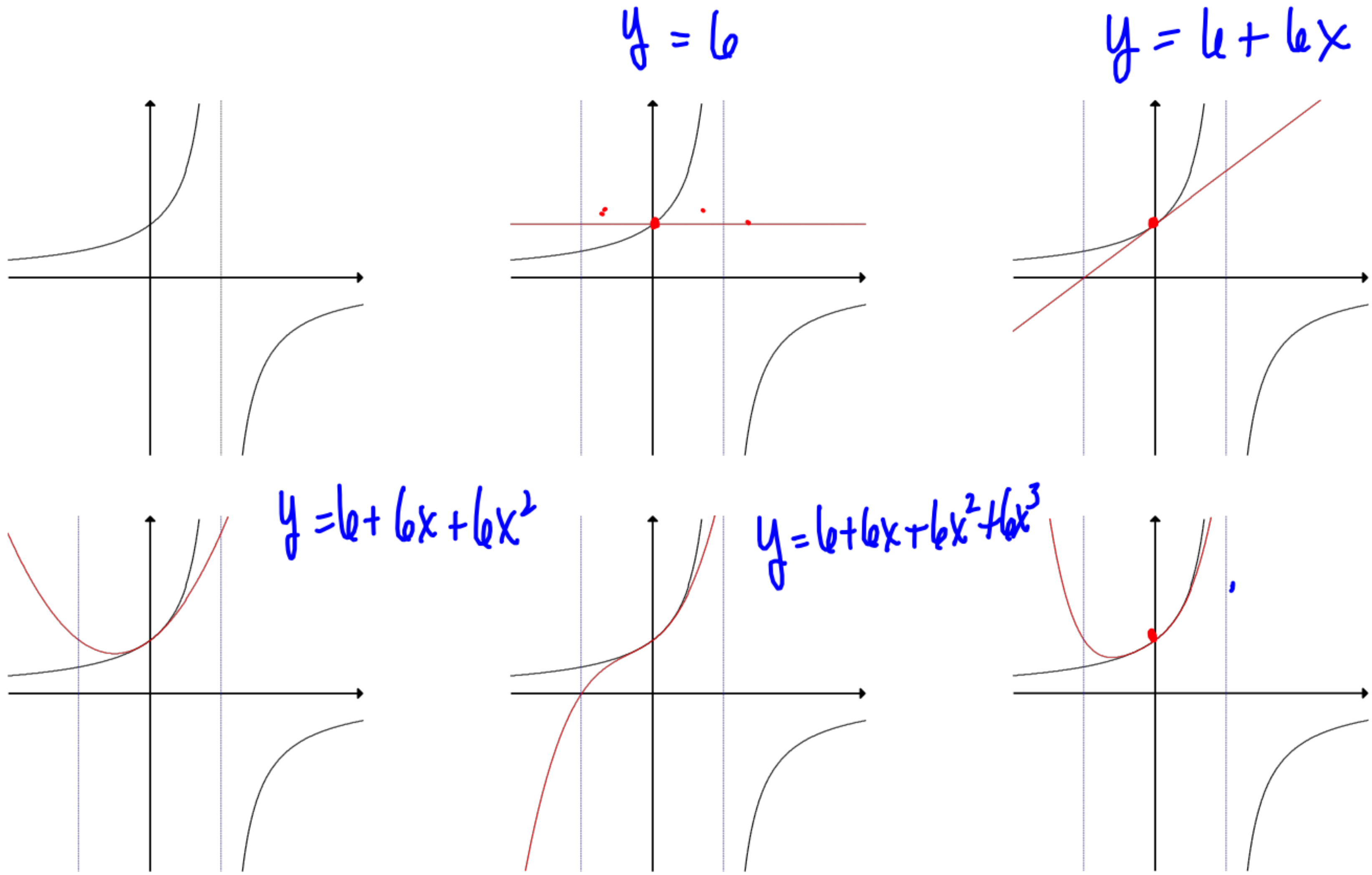
The word series indicates that there is an infinite number of terms.

The word power tells us that each term contains a power of x.

The series is also a geometric series, with |r|=x, so the series will converge for |x|<1.

$-1 < x < 1$

By comparing the graphs of $f(x) = \dfrac{6}{1-x}$ and P(x) with more and more terms, you will see that between –1 and 1 (the interval of convergence), the two graphs converge.



A Power Series (centered at x=0) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

where x is a variable and the c_n's are coefficients.

Note: $\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$ when $|x|<1$

Using this, we can write functions in this form in sigma notation:

$$\frac{1}{1-(?)}$$

Ex: Write $\frac{x^2}{4-x^2}$ as its power series

$$= x^2\left(\frac{1}{4-x^2}\right) = \frac{x^2}{4}\boxed{\left(\frac{1}{1-x^2/4}\right)} \rightarrow \frac{x^2}{4}\boxed{\sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n}$$

$$= \sum_{n=0}^{\infty} \frac{x^2}{4}\cdot\frac{x^{2n}}{4^n} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}}}$$

For a **fixed** x, the series is a series of constants and we can check for convergence/divergence. The series may converge for some values of x and diverge for others.

The sum of the series is

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots + c_nx^n + \ldots$$

whose domain is the set of all x for which the series converges.

f(x) resembles a polynomial, but it has infinitely many terms.

Let $c_n = 1$ for all n, we get the geometric series, centered at $x = 0$,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + \ldots$$

which converges if $|x| < 1$ and diverges if $|x| \geq 1$.

A Power Series (centered at $x=a$) is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \ldots$$

For notation purposes, $(x - a)^0 = 1$ even when $x = a$.

When $x = a$, all the terms are 0 for $n \geq 1$, so the power series always converges when $x = a$.

Ex. For what values of x is the series convergent? $x = 0$

$$\sum_{n=0}^{\infty} n!x^n$$

$\sum \frac{n!}{2^n}$

ex $x = 1/2$

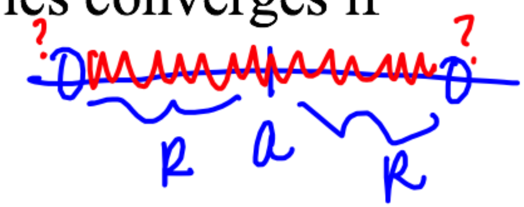
$x = \frac{1}{100000}$ same thing

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only 3 possibilities.

1. The series converges only when $x = a$.

2. The series converges for all x.

3. There is a positive number R such that the series converges if $|x-a| \le R$ and diverges if $|x-a| > R$.



R is the radius of convergence.

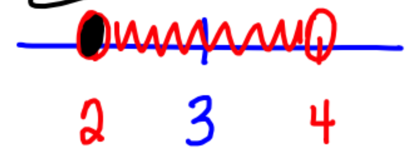
The interval of convergence of a power series is the interval that consists of all values of x for which the series converges absolutely. Check endpoints (endpoints may converge absolutely or conditionally)!

$|n| = n$ since $n > 0$

$|x - a| < \underline{\underline{R}}$

Find the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$.

$\boxed{[2, 4)}$



① take abs. value

② Root test

$$\lim_{n\to\infty} \left(\frac{|x-3|^n}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{|x-3|}{n^{1/n}} \to \frac{|x-3|}{1}$$

conv. if $\frac{|x-3|}{1} < 1$

$|x-3| < 1$ $\boxed{R = 1}$

Radius of conv.

③ CK endpts:

plug endpts into original Σ.

$x = 2$: $\displaystyle\sum \frac{(-1)^n}{n}$ conv. by AST

$x = 4$: $\displaystyle\sum \frac{1^n}{n} = \sum \frac{1}{n}$ div (Harmonic)

Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n2^n}.$$

$$\boxed{(3,7] \quad R=2}$$

abs. value

Root: $\displaystyle \lim_{n\to\infty} \left(\frac{|x-5|^n}{n\,2^n}\right)^{1/n} = \lim_{n\to\infty} \frac{|x-5|}{n^{1/n}\cdot 2} = \frac{|x-5|}{2}$

Conv. when $\dfrac{|x-5|}{2} < 1 \Rightarrow |x-5| < \underset{R}{2}$

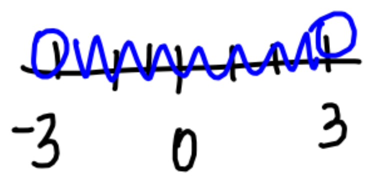
endpts:

$x=3$: $\displaystyle \sum \frac{(-1)^{n+1}(-2)^n}{n\,2^n} = \sum \frac{(-1)^{n+1}}{n}(-1)^n = \sum \frac{(-1)^{2n+1}}{n} = \sum \frac{-1}{n}$ div.

$x=7$: $\displaystyle \sum \frac{(-1)^{n+1}\cdot 2^n}{n\,2^n} = \sum \frac{(-1)^{n+1}}{n}$ conv

Find the radius of convergence and interval of convergence for

$$\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n.$$



$(-3, 3)$

Root: $\displaystyle\lim_{n\to\infty}\left(\frac{|x|^n}{3^n}\right)^{1/n} = \lim_{n\to\infty}\frac{|x|}{3} = \frac{|x|}{3}$

$\displaystyle\frac{|x|}{3} < 1 \Rightarrow |x| < 3$ $R = 3$

endpts:

$x = -3$ $\displaystyle\sum\left(\frac{-3}{3}\right)^n = \sum(-1)^n$ div.

$x = 3$ $\displaystyle\sum\left(\frac{3}{3}\right)^n = \sum 1$ div.

Find the radius of convergence and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}.$$

Find the radius of convergence and interval of convergence for

$$\sum_{n=0}^{\infty} n!(x-3)^n .$$

Power series are continuous functions.

A power series is continuous on its interval of convergence.

If a power series centered at $x = a$ has a radius of convergence $R > 0$, then the power series can be differentiated and integrated on $(a - R, a + R)$, and the new series will converge on $(a - R, a + R)$, and maybe at the endpoints.

pop #11 + 12 = D