Math 1432

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Office Hours:

Mondays 1-2pm,
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Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

Reminder:

Power Series

If $f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n$ has a radius of convergence $R>0$ then,

$$f'(x)=\sum_{n=0}^{\infty} na_n(x-c)^{n-1}=\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

and

$$\int f(x)dx = C+\sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$

both have a radius of convergence of R.

when $x=0$, $f(0)=\ln 2$

Ex: Find a power series for $\boxed{f(x)=\ln(2-x)}$ and determine its interval of convergence.

$$f'(x) = \frac{-1}{2-x}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= -\frac{1}{2}\left(\frac{1}{1-\boxed{x/2}}\right) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$$f'(x) = -\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

$$f(x) = \int -\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}\,dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)2^{n+1}} + C$$

$$f(0) = -\sum_{n=0}^{\infty} 0 + \underline{C = \ln 2}$$

$$f(x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)2^{n+1}} + \ln 2$$

$$\ln 2 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)2^{n+1}}$$

$[-2, 2)$

Ratio: $\lim_{n\to\infty} \frac{|x|^{\cancel{n+2}}}{(n+2)\,2^{\cancel{n+2}}} \cdot \frac{(n+1)\,\cancel{2^{n+1}}}{\cancel{|x|^{n+1}}}$

$$= \lim_{n\to\infty} \frac{|x|}{2}\underbrace{\frac{(n+1)}{(n+2)}}_{\to 1} = \frac{|x|}{2} < 1 \qquad |x| < \underset{R}{\underset{\uparrow}{2}}$$

$x = -2$: $\ln 2 - \sum_{n=0}^{\infty} \frac{\overset{(-1)^{n+1}\cdot 2^{n+1}}{(-2)^{n+1}}}{(n+1)\,2^{n+1}} = \ln 2 - \sum \frac{(-1)^{n+1}}{n+1}$ conv

$x = 2$: $\ln 2 - \sum \frac{2^{n+1}}{(n+1)\,2^{n+1}} = \ln 2 - \sum \frac{1}{n+1}$ div.

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1. Give the 7th degree Taylor polynomial approximation for $f(x) = e^x$ centered at $x = 0$.

a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}$

b. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

c. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

d. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

2. Give the 7th degree Taylor polynomial approximation for $f(x) = \sin(x)$ centered at $x = 0$.

a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}$

b. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

c. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

d. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

3. Give the 7th degree Taylor polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$.

a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}$

b. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

c. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

d. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

4. Give the 7th degree Taylor polynomial approximation for $f(x) = ln(x+1)$ centered at x = 0.

a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}$

b. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

c. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

d. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

5. Give the coefficient of $(x - 1)^3$ for the 8^{th} degree Taylor polynomial approximation to $ln\,(x)$ centered at $x = 1$.

a. $\frac{1}{3}$ b. $\frac{-1}{3}$ c. $\frac{1}{2}$ d. $\frac{-1}{2}$ e. none of these

k	$f^k(x)$	$f^k(1)$	$\frac{f^k(1)}{k!}$	term
0	$\ln x$	0		
1	$1/x$	1		
2	$-1/x^2$	-1		
3	$2/x^3$	2	$\boxed{2/3! = ?}$	

$P_5(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$

Find the Taylor polynomial of degree n = 5 for $f(x) = \ln x$ at c = 1.
Then use $P_5(x)$ to approximate the value of $\boxed{\ln(1.1)}$.

k	$f^k(x)$	$f^k(1)$	$\frac{f^k(1)}{k!}$	term $(x-1)^k$
0	$f(x) = \ln x$	0	0	—
1	$f'(x) = x^{-1}$	1	1	$(x-1)$
2	$f''(x) = -1x^{-2}$	-1	$-1/2$	$-\frac{1}{2}(x-1)^2$
3	$f'''(x) = 2x^{-3}$	2	$2/3! = 1/3$	$\frac{1}{3}(x-1)^3$
4	$f^{(4)}(x) = -6x^{-4}$	-6	$-6/4! = -1/4$	$-\frac{1}{4}(x-1)^4$
5	$f^{(5)}(x) = 24x^{-5}$	24	$24/5! = 1/5$	$\frac{1}{5}(x-1)^5$

$$P_5(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4 + \frac{1}{5}(1.1-1)^5$$

$$\approx .09531033$$

$$\ln(1.1) = .095310179 8$$

Suppose that g is a function which has continuous derivatives, and that

$$g(2) = 3, \quad g'(2) = -4, \qquad g''(2) = 7, \qquad g'''(2) = -5. \leftarrow g^{k}(c)$$

÷ each by k!

Write the Taylor polynomial of degree 3 for g centered at x = 2.

$$\frac{3}{0!}(x-2)^0 + \frac{-4}{1!}(x-2)^1 + \frac{7}{2!}(x-2)^2 + \frac{-5}{3!}(x-2)^3$$

$$3 - 4(x-2) + \frac{7}{2}(x-2)^2 - \frac{5}{6}(x-2)^3$$

Find $P_6(x)$ for $f(x) = x^2\cos(5x)$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$x^2\cos(5x) = x^2\left(\underbrace{1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!}} - \frac{(5x)^6}{6!} + \cdots\right)$$

$$= x^2 - \frac{5^2x^4}{2!} + \frac{5^4x^6}{4!}$$

Find $f^{(15)}(0)$ for $f(x)=e^{x^3}$

$$\boxed{\frac{f^{(15)}(0)}{15!}x^{15}}$$

coeff $\frac{f^{(15)}(0)}{15!}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \boxed{\frac{x^{15}}{5!}}$$

coeff is $1/5!$

$$\frac{f^{(15)}(0)}{15!} = \frac{1}{5!}$$

$$f^{(15)}(0) = \frac{15!}{5!}$$

Lagrange Form of the Remainder
or
Lagrange Error Bound or Taylor's Theorem Remainder

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

$f(x) = P_n(x) + R_n(x)$ so $R_n(x) = f(x) - P_n(x)$

Written in words:

Function = Polynomial + Remainder

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

so

Remainder = Function – Polynomial

Lagrange Formula for Remainder:

Suppose f has n+1 continuous derivatives on an open interval that contains 0. Let x be in that interval and let $P_n(x)$ be the n^{th} Taylor Polynomial for f. Then

some c

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

where c is some number between 0 and x.

If we rewrite Taylor's theorem using the Lagrange formula for the remainder, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}}_{R_n}$$

where c is some number between 0 and x.

If there is a number M so that $\left|f^{(n+1)}(c)\right| \le M$

for all c between 0 and x then $$\left|f(x) - P_n(x)\right| \le \underbrace{\frac{M}{(n+1)!}\left|x\right|^{n+1}}_{\star}$$

$R_n \le$

or

$$\left|R_n(x)\right| \le \left(max\left|f^{(n+1)}(c)\right| \right) \frac{\left|x\right|^{n+1}}{(n+1)!}$$ for c between 0 and x.

We probably will not know the value of c.

Give an error estimate for the approximation of *sin* (x) by $P_9(x)$ for an arbitrary value of x between 0 and $\pi/4$, centered at $x = 0$.

$n=9$

$f(x) = \sin x$ 4 8

$f'(x) = \cos x$ 5 9

$f''(x) = -\sin x$ 6 10

$f'''(x) = -\cos x$ 7

$f^{(4)}(x) = \sin x$

$$R_9 \le \frac{f^{10}(c)}{10!} x^{10} \le \frac{1}{10!} x^{10}$$

$$\le \boxed{\frac{1}{10!}\left(\frac{\pi}{4}\right)^{10}}$$

2.46×10^{-8}

$$f^{10}(x) = -\sin(x)$$

$$\underbrace{|f^{(10)}(c)| \le 1}\leftarrow M$$

$\cos(2x)$

Give an error estimate for the approximation of ~~cos (x)~~ by $P_{10}(x)$ for an arbitrary value of x between 0 and π/4, centered at x = 0.

$f^{(11)}(x)$

$f(x) = \cos 2x$

$f'(x) = -2\sin 2x$

$f''(x) = -4\cos 2x$

$f'''(x) = 8\sin 2x$

$f^{(4)}(x) = 16\cos 2x$

$f^{5}(x) = -32\sin(2x)$

$f^{6}(x) = -64\cos(2x)$

$f^{7}(x) = 128\sin(2x)$

$f^{8}(x) = 256\cos(2x)$

$f^{(9)} = -512\sin(2x)$

$f^{(10)}(x) = -1024\cos(2x)$

$f^{(11)}(x) = 2048\sin(2x)$

$|f^{(11)}(c)| \le 2048 = M$

$$R_{10} \le \frac{2048}{11!}\left(\frac{\pi}{4}\right)^{11}$$

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6. Assume that *f(x)* is a function such that $\left|f^{(10)}(x)\right| < 15$ for all x in the interval (0,1). What is the max possible error for the ninth degree Taylor polynomial centered at 0 for this function when approximating *f*(1)?

P_9

a. 15
b. 15/9!
c. 15/10!
d. 1
~~**e.**~~ none of these

$R_9 \leq$