Math 1432

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Office Hours:

Mondays 1-2pm,
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(also available by appointment)

Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

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1. Compute $\lim_{x\to 0} \dfrac{x-\sin x}{x-\tan x}$.

2. Compute $\lim_{x\to 0} \dfrac{e^{x}-e^{-2x}}{2\sin x}$.

3. Compute $\lim_{x\to \infty} \left(\arctan(x)\right)$.

Improper Integrals

The definition of the definite integral $\int_a^b f(x)\,dx$ requires that [a, b] be finite and that f(x) be bounded on [a, b].

Also, the Fundamental Theorem of Calculus requires that f be continuous on [a, b].

If one or both of the limits of integration are infinite or if f has a finite number of infinite discontinuities on [a, b], then the integral is called an improper integral.

Types of improper integrals:

A. (one or both bounds are infinite)

$\int_1^\infty \frac{dx}{x}$, $\int_{-\infty}^1 \frac{3dx}{x^4+5}$ and $\int_{-\infty}^\infty \frac{dx}{x^2+1}$ are improper because one or both bounds are infinite.

B. (infinite discontinuity at a boundary)

$\int_1^5 \frac{dx}{\sqrt{x-1}}$ is improper because $f(x) = \frac{1}{\sqrt{x-1}}$ has an infinite discontinuity at $x = 1$.

C. (infinite discontinuity in the interior)

$\int_{-2}^2 \frac{dx}{(x+1)^2}$ is improper because $f(x) = \frac{1}{(x+1)^2}$ has an infinite discontinuity at $x = -1$, and -1 is between -2 and 2.

For the first type of improper integrals:

1) If f is continuous on $\left[a, \infty\right)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2) If f is continuous on $\left(-\infty, b\right]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3) If f is continuous on $\left(-\infty, \infty\right)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx.$$

If the limit exists, then the improper integral is said to converge. Otherwise, it diverges.

Examples for the first type of improper integral.

1. $\int_{1}^{\infty} \frac{dx}{x}$

2. $\int_{2}^{\infty} e^{-x}\, dx$

3. $\int_0^\infty \frac{1}{x^2+1}\,dx$

4. $\int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{2x}}\, dx$

The second and third type of improper integral:

1. If f is continuous on $[a,b)$ but has an infinite discontinuity at b, then
$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

2. If f is continuous on $(a,b]$ but has an infinite discontinuity at a, then
$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

3. If f is continuous on $[a, b]$ except for some c in (a, b) at which f has an infinite discontinuity, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$
provided **both** integrals on the right converge. If either integral on the right diverges, we say that the integral on the left diverges.

Examples for the second type of improper integral.

1. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

2. $\int_0^2 \frac{dx}{x^3}$

3. $\displaystyle\int_{0}^{27} \frac{dx}{\sqrt[3]{27-x}}$

4. $\int_{1}^{4} \frac{dx}{x-2}$

Important examples:

$$\int_1^\infty \frac{dx}{x^p} \qquad p = 1$$

$$\int_1^\infty \frac{dx}{x^p} \qquad p > 1$$

$$\int_1^\infty \frac{dx}{x^p} \qquad 0 < p < 1$$

$$\int_1^\infty \frac{dx}{x^p}$$

Diverges for $p \le 1$
Converges for $p > 1$

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Which of the following are improper integrals?