The Integral Test and Approximating Sums

Thm: The Integral Test

If f is positive, continuous and decreasing for x ≥ 1 and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad and \quad \int_0^{\infty} f(x)dx$$

either both converge or both diverge. If the series converges to S, then the remainder $R_N = S - S_N$ is bounded by

$$0 \le R_N \le \int_N^{\infty} f(x)dx$$

In other words,

$$0 \le S - S_N \le \int_N^{\infty} f(x)dx \quad => \quad S_N \le S \le S_N + \int_N^{\infty} f(x)dx$$

NOTES:

1. The Integral Test is for *positive series* only; that is when the series has all positive terms.
2. In the application of this test, neither the series nor the integral has to start with 1.
3. This test *does not* give the sum of the series. This test states that the convergence of one implies the convergence of the other. (And the divergence of one implies the divergence of the other.)

Examples:

1) Using the integral test:

a)

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$f(x) = \frac{x}{x^2+1} \text{ is positive, decreasing, and continuous} => \text{can use integral test}$$

$$\int_0^{\infty} \frac{x}{x^2+1}dx = \frac{1}{2}\int_0^{\infty} \frac{2x}{x^2+1}dx$$

$$= \frac{1}{2}\lim_{n->\infty}[\ln(x^2+1)]_1^n = \infty$$

$=>$ *diverges*

b)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

$$f(x) = \frac{1}{x^2+1} \text{ is positive, decreasing, and continuous} => \text{can use integral test}$$

$$\int_0^{\infty} \frac{1}{x^2+1}dx = \lim_{n->\infty}[\arctan x]_1^n = \lim_{n->\infty}[\arctan n - \arctan 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$=>$ *converges*