Trigonometric formulas

$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$

$\sin(A+B) = \sin A\cos B + \sin B\cos A$	$\sin(A-B) = \sin A\cos B - \sin B\cos A$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$	$\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$
$\csc\theta = \dfrac{1}{\sin\theta}$	$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$	$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

Differentiation formulas

$\dfrac{d}{dx}(x^n) = nx^{n-1}$	$\dfrac{d}{dx}(fg) = fg' + gf'$	$\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{gf' - fg'}{g^2}$
$\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$	$\dfrac{d}{dx}(\sin x) = \cos x$	$\dfrac{d}{dx}(\cos x) = -\sin x$
$\dfrac{d}{dx}(\tan x) = \sec^2 x$	$\dfrac{d}{dx}(\cot x) = -\csc^2 x$	$\dfrac{d}{dx}(\sec x) = \sec x\tan x$
$\dfrac{d}{dx}(\csc x) = -\csc x\cot x$	$\dfrac{d}{dx}(e^x) = e^x$	$\dfrac{d}{dx}(a^x) = a^x\ln a$
$\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$	$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$
$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$		

Integration formulas

$\int a\,dx = ax + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	$\int \frac{1}{x} dx = \ln x + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$	$\int \ln x\,dx = x \ln x - x + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \cot x\,dx = \ln(\sin x) + C$	$\int \sec x\,dx = \ln(\sec x + \tan x) + C$	$\int \csc x\,dx = \ln(\csc x - \cot x) + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \csc x \cot x\,dx = -\csc x + C$	$\int \tan^2 x\,dx = \tan x - x + C$	$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \tan x\,dx = \ln(\sec x) + C$ or $-\ln(\cos x) + C$		

If the Integral Involves	Then Substitute	And Use the Identity
$a^2 - u^2$	$u = a \sin \theta$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + u^2$	$u = a \tan \theta$	$1 + \tan^2\theta = \sec^2\theta$
$u^2 - a^2$	$u = a \sec \theta$	$\sec^2\theta - 1 = \tan^2\theta$

$y = D + A\sin B(x - C)$ **A is amplitude B is the affect on the period (stretch or shrink)**

C is vertical shift (left/right) and D is horizontal shift (up/down)

Limits:

$\lim_{x->0} \frac{\sin x}{x} = 1$ $\quad$ $\lim_{x->\infty} \frac{\sin x}{x} = 0$ $\quad$ $\lim_{x->0} \frac{1 - \cos x}{x} = 0$

Exponential Growth and Decay

$$y = Ce^{kt}$$

Rate of Change of a variable y is proportional to the value of y

$$\frac{dy}{dx} = ky \quad or \quad y' = ky$$

Formulas and theorems

1. **A function *y=f(x)* is continuous at *x=a* if**
   i. *f(a)* exists
   ii. $\lim_{x \to a} f(x)$ exists, and
   iii. $\lim_{x \to a} f(x) = f(a)$
2. **Even and odd functions**
   1. A function *y = f(x)* is even if *f(-x) = f(x)* for every *x* in the function's domain. Every even function is symmetric about the y-axis.
   2. A function *y = f(x)* is odd if *f(-x) = –f(x)* for every *x* in the function's domain. Every odd function is symmetric about the origin.
3. **Horizontal and vertical asymptotes**
   1. A line *y = b* is a horizontal asymptote of the graph of *y = f(x)* if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.
   2. A line *x =a* is a vertical asymptote of the graph of *y =f(x)* if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

4. **Definition of a derivative**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5. **To find the maximum and minimum values of a function y = f(x), locate**

   1. . the points where *f'(x)* is zero or where *f'(x)* fails to exist
   2. the end points, if any, on the domain of *f(x)*.

      Note: These are the only candidates for the value of *x* where *f(x)* may have a maximum or a minimum

6. **Let f be differentiable for $a < x < b$ and continuous for $a \le x \le b$.**
   a. If $f'(x) > 0$ for every x in (a,b), then f is increasing on $[a,b]$.
   b. If $f'(x) < 0$ for every x in (a,b), then f is decreasing on $[a,b]$.
7. **Suppose that f''(x) exists on the interval (a,b).**
   a. If $f''(x) > 0$ in (a,b), then f is concave upward in (a,b).
   b. If $f''(x) < 0$ in (a,b), then f is concave downward in (a,b).

   To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ or where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

8. **Mean value theorem**

   If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

9. **Continuity**

   If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does not imply differentiability.

10. **L'Hôpital's rule**

   If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

11. **Area between curves**

   If f and g are continuous functions such that $f(x) \ge g(x)$ on $[a,b]$, then the area between the curves is $\int_a^b (f(x) - g(x))dx$.

12. **Inverse functions**
   a. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g, and, $g(f(x)) = x$, for every x in the domain of f, then, f and g are inverse functions of each other.
   b. A function f has an inverse if and only if no horizontal line intersects its graph more than once.
   c. If f is either increasing or decreasing in an interval, then f has an inverse.
   d. If f is differentiable at every point on an interval I, and $f'(x) \ne 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and $g'(f(x)) = \frac{1}{f'(x)}$.

13. **Properties of $y = e^x$**

a. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.

b. The domain is the set of all real numbers, $-\infty < x < \infty$.

c. The range is the set of all positive numbers, $y > 0$.

d. $\frac{d}{dx}(e^x) = e^x$

e. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

14. **Properties of $y = \ln x$**

a. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.

b. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.

c. $y = \ln x$ is continuous and increasing everywhere on its domain.

d. $\ln(ab) = \ln a + \ln b$.

e. $\ln(a / b) = \ln a - \ln b$.

f. $\ln a^r = r \ln a$.

15. **Fundamental theorem of calculus**

$\int_a^b f(x)dx = F(b) - F(a)$, where F'(x) = f(x), or $\frac{d}{dx}\int_a^x f(x)dx = f(x)$.

16. **Volumes of solids of revolution**

a. Let f be nonnegative and continuous on $[a,b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis, and the sides by the lines $x = a$ and $x = b$.

b. When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \int_a^b \pi(f(x))^2 dx$.

c. When R is revolved about the y-axis, it generates a solid whose volume $V = \int_a^b 2\pi \cdot x \cdot f(x)dx$.

17. **Particles moving along a line**

a. If a particle moving along a straight line has a positive function $x(t)$, then its instantaneous velocity $v(t) = x'(t)$ and its acceleration $a(t) = v'(t)$.

b. $v(t) = \int a(t)dt$ and $x(t) = \int v(t)dt$.

18. **Average y-value**

The average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

Summary of Convergence Tests for Series

Test	Series	Convergence or Divergence	Comments
n^{th} term test (or the zero test)	$\sum a_n$	Diverges if $\lim\limits_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim\limits_{n\to\infty} a_n = 0$.
Geometric series	$\sum\limits_{n=0}^{\infty} ax^n \left(\text{or} \sum\limits_{n=1}^{\infty} ax^{n-1}\right)$	Converges to $\dfrac{a}{1-x}$ only if $\lvert x\rvert < 1$ Diverges if $\lvert x\rvert \geq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to ax^n.
p-series	$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$	Converges if $p > 1$ Diverges if $p \leq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to $\dfrac{1}{n^p}$.
Integral	$\sum\limits_{n=c}^{\infty} a_n \quad (c \geq 0)$ $a_n = f(n)$ for all n	Converges if $\int_c^{\infty} f(x)\,dx$ converges Diverges if $\int_c^{\infty} f(x)\,dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing and readily integrable for $x \geq c$.
Comparison	$\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ for all n	$\sum b_n$ converges $\Longrightarrow \sum a_n$ converges $\sum a_n$ diverges $\Longrightarrow \sum b_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p-series.
Limit Comparison*	$\sum a_n$ and $\sum b_n$ with $a_n, b_n > 0$ for all n and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$	$\sum b_n$ converges $\Longrightarrow \sum a_n$ converges $\sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p-series. To find b_n consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$ with $\lim\limits_{n\to\infty} \dfrac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Inconclusive if $L = 1$. Useful if a_n involves factorials or n^{th} powers.
Root*	$\sum a_n$ with $\lim\limits_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Test is inconclusive if $L = 1$. Useful if a_n involves n^{th} powers.
Absolute Value $\sum \lvert a_n\rvert$	$\sum a_n$	$\sum \lvert a_n\rvert$ converges $\Longrightarrow \sum a_n$ converges	Useful for series containing both positive and negative terms.
Alternating series	$\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n \quad (a_n > 0)$	Converges if $0 < a_{n+1} < a_n$ for all n and $\lim\limits_{n\to\infty} a_n = 0$	Applicable only to series with alternating terms.

Sequence and Series Summary Formulas

1. If a sequence $\{a_n\}$ has a limit L, that is, $\lim_{n\to\infty} a_n = L$, then the sequence is said to <u>converge</u> to L. If there is no limit, the series <u>diverges</u>. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that

   $\lim_{n\to\infty} \frac{\ln n}{n} = 0$; $\lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1$; $\lim_{n\to\infty} \sqrt[n]{n} = 1$; $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. These limits are useful and arise frequently.

2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \ge 1$ and $a \ne 0$.

3. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

4. <u>Limit Comparison Test</u>: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of nonnegative terms, with $a_n \ne 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty} \frac{b_n}{a_n} = c > 0$. Then the two series either both converge or both diverge.

5. <u>Alternating Series</u>: Let $\sum_{n=1}^{\infty} a_n$ be a series such that
   - i) the series is alternating
   - ii) $|a_{n+1}| \le |a_n|$ for all n, and
   - iii) $\lim_{n\to\infty} a_n = 0$

   Then the series converges.

6. A series $\sum a_n$ is <u>absolutely convergent</u> if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is <u>conditionally convergent</u>. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

7. Comparison Test: If $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
8. Integral Test: If $f(x)$ is a positive, continuous, and decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral $\int_1^{\infty} f(x)\,dx$ converges. If the improper integral $\int_1^{\infty} f(x)\,dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.
9. Ratio Test: Let $\sum a_n$ be a series with nonzero terms.
   - i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely.
   - ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series is divergent.
   - iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive (and another test must be used).
10. Power Series: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots \quad \text{or}$$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants. The set of all numbers x for which the power series converges is called the interval of convergence.
11. Taylor Series: Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + ..$$

The remaining terms after the term containing the nth derivative can be expressed as a remainder to Taylor's Theorem:

$$f(x) = f(a) + \sum_{1}^{n} f^{(n)}(a)(x-a)^n + R_n(x) \quad \text{where } R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Lagrange's form of the remainder: $R_n x = \dfrac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$, where $a < c < x$. The series will converge for all values of x for which the remainder goes to zero.

12. <u>Frequently Used Series</u>

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + ... = \sum_{n=0}^{\infty} x^n \text{ , } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - ... + (-x)^n + ... = \sum_{n=0}^{\infty} (-1)^n x^n \text{ , } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{, } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{, } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^n \frac{x^{2n}}{(2n)!} + ... = \sum_{n=0}^{\infty} \frac{(-1) x^{2n}}{(2n)!} \text{, } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1} \frac{x^n}{n} + ... = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \text{, } -1 < x \le 1$$

$$Arc\tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - ... + (-1)^n \frac{x^{2n+1}}{2n+1} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{, } |x| \le 1$$

Indeterminate Form:

$\frac{0}{0}, \quad \frac{\infty}{\infty} \quad \Rightarrow$ Apply L'Hopital Directly

$0 \cdot \infty \quad \Rightarrow$ Rewrite as either $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Then apply L'Hopital

$1^{\infty}, 0^{0}, \infty^{0} \quad \Rightarrow$

1. Consider the limit of the ln of the function.
2. Use laws of logs to rewrite in the form $0 \cdot \infty$.
3. Rewrite as either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
4. Apply L'Hopital.
5. Exponentiate your answer.

$\infty - \infty \quad \Rightarrow$ Try to rewrite so that you can use one of the previous forms.

To convert polar coordinates into rectangular coordinates, we use the basic relations

$x = r \cos \theta, \qquad y = r \sin \theta$

Converting in the opposite direction we use

$r^2 = x^2 + y^2, \quad \tan \theta = y/x$ if $x \neq 0$

What does the graph look like?

$r = a$ $\Rightarrow$ Circle

$r = \theta$ $\Rightarrow$ Line

$r = a + b \sin\theta$ OR $r = a + b \cos\theta$

$a > b$ $\Rightarrow$ Dimpled Limacon

$a < b$ $\Rightarrow$ Limacon with an inner loop

$a = b$ $\Rightarrow$ Cardiod

$r = a \cos n\theta$ OR $r = a \sin n\theta$

n even ($n \geq 2$) $\Rightarrow$ Rose with 2n petals.

n odd ($n \geq 3$) $\Rightarrow$ Rose with n petals.