HOMEWORK 3 (Answer Key)

2. (b) $Q=0,\ u_x(0)=0,\ u(L)=T$: $u_{xx}=0 \Rightarrow u(x)=c_1x+c_2$
$u_x(0)=c_1=0 \Rightarrow u(x)=c_2$
$u(L)=T=c_2 \Rightarrow \boxed{u(x)=T}$

(a), (d), (f), (h) → textbook
(e), (g) → solved in class

3. $E(t)=c\rho\int_0^L u(x,t)A\,dx \Rightarrow \frac{dE}{dt}=c\rho\int u_t A\,dx \overset{\text{PDE}}{=} c\rho\int_0^L K_0 u_{xx}A\,dx$
$\overset{\text{integrate}}{=} c\rho K_0 A\left[u_x\right]_0^L =$
(solved in class)
$= c\rho K_0 A\big[\underbrace{u_x(L,t)}_{=0}-\underbrace{u_x(0,t)}_{=0}\big]=0$

4. (a) $u_{xx}+1=0 \Rightarrow u_{xx}=-1 \Rightarrow u_x=-x+C_1 \Rightarrow u(x)=-\frac{x^2}{2}+C_1x+C_2$
$u_x(0,t)=1 \Rightarrow u_x(0,t)=0+C_1=1 \Rightarrow \boxed{C_1=1}$
$u_x(L,t)=\beta \Rightarrow u_x(L,t)=1-L \Rightarrow \beta$ must be equal to $1-L$. $\boxed{\beta=1-L}$
FOR A SOLUTION TO EXIST.

(b) $u_{xx}=0 \Rightarrow u_x=C_1 \Rightarrow u=C_1x+C_2$
$u_x(0,t)=1 \Rightarrow \boxed{C_1=1} \Rightarrow u=C_1x+C_2=x+C_2$
$u_x(L,t)=\beta \Rightarrow \beta$ must be equal to 1 : $\boxed{\beta=1}$ for a solution to exist

Solution: $\boxed{u^e(x)=x+C_2}$

No sources, flux of the left is equal to flux on the right
⇒ energy conserved. Determine C_2 from initial condition
Energy of initial data = energy of equilibrium

$E(0)=\cancel{c\rho}\int_0^L f(x)A\,dx = c\rho\int_0^L u^e(x)A\,dx = \cancel{c\rho}\int_0^L(x+C_2)A\,dx$

$\Rightarrow \int_0^L f(x)\,dx = \int_0^L(x+C_2)\,dx = \left[\frac{x^2}{2}+C_2x\right]_0^L = \frac{L^2}{2}+C_2L \Rightarrow$

$C_2=\frac{1}{L}\left[\int_0^L f(x)dx-\frac{L^2}{2}\right] \Rightarrow \boxed{u^e(x)=x+\frac{1}{L}\left[\int_0^L f(x)dx-\frac{L^2}{2}\right]}$

(c) $u_{xx} + x - \beta = 0 \Rightarrow u_{xx} = \beta - x \Rightarrow u_x = \beta x - \frac{x^2}{2} + c_1 \Rightarrow$

$$\boxed{u(x) = \beta \frac{x^2}{2} - \frac{x^3}{6} + c_1 x + c_2}$$

$u_x(0,t) = 0 \Rightarrow u_x(0) = c_1 = 0 \Rightarrow \boxed{c_1 = 0}$

$u_x(L,t) = 0 \Rightarrow u_x(L) = \beta L - \frac{L^2}{2} + 0 = 0 \Rightarrow \boxed{\beta = \frac{L}{2}}$

Equilibrium solution exists if $\beta = \frac{L}{2}$. In that case the equilibrium solution is given by

$$u^e(x) = \frac{L}{2}\frac{x^2}{2} - \frac{x^3}{6} + c_2$$

Is it possible to determine c_2?

~~Insulated end points ⇒ ...~~

Is the energy conserved in this problem?

Answer: calculate $\frac{dE}{dt}$ and check if it is 0.

$$\frac{dE}{dt} = c\rho \int_0^L u_t A\,dx = c\rho A \int_0^L \underbrace{(u_{xx} + x - \tfrac{L}{2})}_{\text{PDE: } u_t = u_{xx} + x - \frac{L}{2}} dx =$$

$$= c\rho A \left[u_x + \frac{x^2}{2} - \frac{L}{2}x \right]_0^L = c\rho A \Big\{ \underbrace{[u_x]_0^L}_{=0 \text{ (insulated ends)}} + \frac{L^2}{2} - \frac{L^2}{2} \Big\} = 0$$

Since energy is conserved, $E(0) = E(t)$, for all t

Thus, the total energy of u^e is equal to the energy of initial data $u(x,0) = f(x)$. From here we determine c_2:

$$t) = \rho c \int_0^L u^e(x) A\,dx = \rho c A \int_0^L \left(\frac{L}{4}x^2 - \frac{x^3}{6} + c_2\right) dx = E(0) = c\rho A \int_0^L f(x)\,dx$$

$$\Rightarrow \int_0^L \left(\frac{L}{4}x^2 - \frac{x^3}{6} + c_2\right) dx = \int_0^L f(x)\,dx$$

$$\frac{L}{4}\frac{L^3}{3} - \frac{L^4}{24} + c_2 L = \int_0^L f(x)\,dx \Rightarrow \boxed{c_2 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^4}{24}\right)}$$

$$\boxed{u^e(x) = \frac{L}{4}x^2 - \frac{x^3}{6} + \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^4}{24}\right)}$$

HOMEWORK 4

1. L_1, L_2 linear , $L(u) = c_1 L_1(u) + c_2 L_2(u)$ $\left(\begin{array}{l}\text{LIN. COMB OF} \\ \text{LIN. OPERATORS}\end{array}\right)$

Show that L is linear. More precisely, show that

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L(u_1) + \alpha_2 L(u_2).$$

<u>Proof:</u> $L(\alpha_1 u_1 + \alpha_2 u_2) =$ by definition $= c_1 L_1(\alpha_1 u_1 + \alpha_2 u_2) + c_2 L_2(\alpha_1 u_1 + \alpha_2 u_2)$

$= (L_1 \& L_2 \text{ linear}) = c_1\left[\alpha_1 L_1(u_1) + \alpha_2 L_1(u_2)\right] + c_2\left[\alpha_1 L_2(u_1) + \alpha_2 L_2(u_2)\right]$

$= \text{regroup terms} = \alpha_1 \underbrace{\left[c_1 L_1(u_1) + c_2 L_2(u_1)\right]} + \alpha_2 \underbrace{\left[c_1 L_1(u_2) + c_2 L_2(u_2)\right]}$

$= \text{definition of } L = \alpha_1 L(u_1) + \alpha_2 L(u_2)$ //

2. Show that $L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$ for $L(u) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right)$

or: show that

$$\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial}{\partial x}(c_1 u_1 + c_2 u_2)\right) = c_1 \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u_1}{\partial x}\right) + c_2 \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u_2}{\partial x}\right).$$

Go step by step to show this.

3. (a) Suppose $Q = \alpha(x,t)u + \beta(x,t)$. Define $L(u) = \frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} + Q$.

Show that $L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$.

(b) Show that $L(0) = 0$ if $\beta = 0$.

4. In class.