Homework 11: March 28, 2017

Sturm-Liouville Eigenvalue Problems

1. Consider the eigenvalue problem

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi, \ x \in (0, L) \\ \phi(0) &= 0, \\ \phi(L) &= 0. \end{aligned} \tag{1}$$

From the Rayleigh quotient formula

$$\lambda = \frac{[-p\phi \ (d\phi/dx)]|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]dx}{\int_a^b \phi^2 \sigma dx}$$

which holds for Sturm-Liouville eigenvalue problems, show that eigenvalue problem (1) must have eigenvalues λ which are all positive.

2. Solve the non-constant coefficient heat equation

$$\frac{\partial u}{\partial t} = 3\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad r \in (1, 2), t > 0$$

describing heat flow in a circular annulus bounded by the inner circle of radius $r = 1$, and the by the outer circle of radius $r = 2$. The annulus material has thermal diffusivity $k = 3$. Suppose that the initial temperature of the annulus is

$$u(r, 0) = f(r),$$

and that the inner and outer circular boundary are insulated, i.e.,

$$\frac{\partial u}{\partial r}(1, t) = 0, \quad \frac{\partial u}{\partial r}(2, t) = 0.$$

You may assume that the corresponding eigenfunctions, denoted $\phi_n(r)$, are known and are complete. Write the formula for the solution in terms of the general Fouriser series, and specify how would you calculate the coefficients in the generalized Fourier series.