Homework 12: April 5, 2017

Laplace's Equation

1. Solve:

$$\begin{aligned} \triangle u &= 0,\ x \in (0, L), y \in (0, H) \\ u_x(0, y) &= 0,\ y \in (0, H) \\ u_x(L, y) &= 0,\ y \in (0, H) \\ u(x, 0) &= 0,\ x \in (0, L) \\ u(x, H) &= f(x),\ x \in (0, L). \end{aligned}$$

Answer: $u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{L} \cos \frac{n\pi x}{L}$, where the coefficients are given by: $A_0 = \frac{1}{HL}\int_0^L f(x)dx$, and $A_n = \frac{1}{\sinh \frac{n\pi H}{L}} \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$.

2. Solve:

$$\begin{aligned} \triangle u &= 0,\ x \in (0, L), y \in (0, H) \\ u_x(0, y) &= 0,\ y \in (0, H) \\ u(L, y) &= g(y),\ y \in (0, H) \\ u(x, 0) &= 0,\ x \in (0, L) \\ u(x, H) &= 0,\ x \in (0, L). \end{aligned}$$

Answer: $u(x, y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{H} \sin \frac{n\pi y}{H}$, where the coefficients are given by: $A_n = \frac{1}{\cosh \frac{n\pi L}{H}} \frac{2}{H}\int_0^L g(y)\sin\frac{n\pi y}{H}dy$.

3. Solve Laplace's equation inside a quarter circle of radius 1 ($0 \le \theta \le \pi/2$, $0 < r \le 1$) subject to the boundary conditions listed below:

$$\begin{aligned} \triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0, 0 \le \theta \le \pi/2, 0 < r \le 1 \\ \frac{\partial u}{\partial \theta}(r, 0) &= 0,\ r \in (0, 1) \\ u(r, \pi/2) &= 0,\ r \in (0, 1) \\ u(1, \theta) &= f(\theta),\ \theta \in (0, \pi/2) \end{aligned}$$

Answer: $u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos(2n-1)\theta$.

Laplace's equation in a circle will be covered in class next week; you may want to wait to solve this problem until then.