NAME (please print):

Midterm Exam 1

1. (15 pts) Determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

$$u_t = u_{xx} + 1, \quad u(x,0) = f(x), \quad u_x(0,t) = 1, \quad u_x(L,t) = \beta.$$

2. (10 pts) Is the following operator

$$L(u) = 10\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

linear? Prove your statement.

3. Consider the heat equation $u_t = 10u_{xx}$, for $x \in (0,1)$, $t > 0$, subject to the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$, and the initial condition

$$u(x,0) = 4\cos\frac{3\pi x}{L}\sin\frac{\pi x}{L}$$

describing heat transfer in a 1D rod.
(a) (10 pts) Write and solve the ODEs for the calculation of separated solutions. What are all the eigenfunctions and eigenvalues associated with this problem?
(b) (5 pts) Is there an eigenvalue that is equal to zero?
(c) (5 pts) Write the general solution satisfying the PDE and the boundary conditions.
(d) (10 pts) Write the Fourier series for the initial datum $u(x,0) = 4\cos\frac{3\pi x}{L}\sin\frac{\pi x}{L}$ in terms of the eigenfunctions found in part (a).
(e) (5 pts) Write the particular solution satisfying the PDE, boundary data, and the initial data written above.
(f) (5 pts) What is the limit as $t \to \infty$ of the solution $u(x,t)$?

Hint: Use: $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$.

4. (a) (20 pts) Solve the heat equation $u_t = ku_{xx}$ for $x \in (0,L), t > 0$ with the boundary conditions

$$u_x(0,t) = 0, \quad u_x(L,t) = 0,$$

and the following initial data:

$$u(x,0) = 1 - \sin^2\frac{4\pi x}{L} + \cos\frac{\pi x}{L}.$$

(b) (5 pts) What is the limit as $t \to \infty$ of the solution found in problem 4 above?
(c) (10 pts) Calculate the total energy in the rod as a function of time. Is the energy conserved?

Hint: Use the following trigonometric identity: $\cos 2\alpha = 1 - 2\sin^2\alpha$.

MIDTERM 1

1 $\begin{cases} u_t = u_{xx}+1 \\ u_x(0,t)=1 \\ u_x(L,t)=\beta \\ u(x,0)=f(x) \end{cases}$

EQ SOLUTION u^E solves $u_{xx}+1=0 \Rightarrow$
$\Rightarrow u_x = -x + C_1 \Rightarrow \boxed{u^E(x) = -\frac{x^2}{2}+C_1x+C_2}$

$u_x(0,t)=1 \Rightarrow \boxed{C_1=1}$

$u_x(L,t)=\beta \Rightarrow \boxed{-L+C_1=\beta}$. From here, in order to have a solution, $C_1=\beta+L$ must be equal to 1. Thus, $\boxed{1=\beta+L}$. From here we calculate $\boxed{\beta=-L+1}$.

Thus, a solution exists if $\boxed{\beta=-L+1}$.

Calculate equilibrium solution: use total energy to find C_2.

We have $u^E(x) = -\frac{x^2}{2}+x+C_2$.

First check if the total energy is conserved:

$$\frac{dE}{dt} = \rho c A\int_0^L u_t\,dx = \rho c A\int_0^L (u_{xx}+1)\,dx = \rho c A\left\{[u_x]_{x=0}^{L}+L\right\} = \rho c A\{\cancel{1}-\cancel{L}-\cancel{1}+\cancel{L}\}$$

Thus, $\frac{dE}{dt}=0$! So, total energy is conserved and so $E(t)=E(0)$ for all t! Thus, the total energy of the equilibrium state = total energy of the initial data

$$\cancel{c\rho A}\int_0^L u^E(x)\,dx = \cancel{c\rho A}\int_0^L f(x)\,dx$$

$$\Rightarrow \int_0^L\left(-\frac{x^2}{2}+x+C_2\right)dx = \int_0^L f(x)\,dx \Rightarrow -\frac{L^3}{6}+\frac{L^2}{2}+C_2L = \int_0^L f(x)\,dx \Rightarrow$$

$$C_2 = \frac{1}{L}\left\{\int_0^L f(x)\,dx + \frac{L^3}{6}-\frac{L^2}{2}\right\}$$

Finally: $\boxed{u^E(x) = -\frac{x^2}{2}+x+\frac{1}{L}\int_0^L f(x)\,dx+\frac{L^2}{6}-\frac{L}{2}}$

2 $L(u) = 10\,u_{xx}+u_x$ is a Linear operator.

Proof: $L(\alpha_1u_1+\alpha_2u_2) = 10(\alpha_1u_1+\alpha_2u_2)_{xx}+(\alpha_1u_1+\alpha_2u_2)_x =$

$=$ (by addition property of partial derivatives) $= 10\{(\alpha_1u_1)_{xx}+(\alpha_2u_2)_{xx}\}+$

$+ (\alpha_1u_1)_x+(\alpha_2u_2)_x = ($~~[illegible]~~

$=$ (use the property: how to differentiate a function times a constant) $= 10\{\alpha_1(u_1)_{xx}+\alpha_2(u_2)_{xx}\}+$

$+\alpha_1(u_1)_x+\alpha_2(u_2)_x$ = combine = $\alpha_1(10(u_1)_{xx}+(u_1)_x)+$

$+\alpha_2(10(u_2)_{xx}+(u_2)_x) = \alpha_1L(u_1)+\alpha_2L(u_2)$ //

③ $u_t = 10\,u_{xx}\ ,\ x\in(0,1)\ ,\ t>0$ ②

$u(0,t)=0$

$u(1,t)=0$

$u(x,0) = 4\cos\frac{3\pi x}{L}\sin\frac{\pi x}{L} = 2\left(\sin\frac{4\pi x}{L} + \sin\left(-\frac{2\pi x}{L}\right)\right)$

$= 2\left(\sin\frac{4\pi x}{L} - \sin\frac{\pi x}{L}\right)$

(a) $u(x,t) = \phi(x)G(t) \Rightarrow \quad G'(t)\phi(x) = 10\,\phi''(x)G(t) \Rightarrow$

$\frac{G'(t)}{10\,G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda \Rightarrow \begin{cases} G'(t) = -10\lambda G(t) \\ \phi''(x) + \lambda\phi(x) = 0 \end{cases}$

Solutions: $\boxed{G(t) = Ce^{-10\lambda t}}$ $\boxed{\phi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x}$

Boundary data: $\phi(0)=0 \Rightarrow A=0,\ \phi(L)=0 \Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow$ $\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$

$\boxed{u_n(x,t) = e^{-10\left(\frac{\pi n}{L}\right)^2 t}\sin\frac{n\pi x}{L} = e^{-10(n\pi)^2 t}\sin(n\pi x)}$, $n=1,2,\dots$

(b) No, there are no zero eigenvalues

(c) $\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-10(n\pi)^2 t}}$

(d) $u(x,0) = 4\cos\frac{3\pi x}{L}\sin\frac{\pi x}{L} = 2\sin\frac{4\pi x}{L} - 2\sin\frac{\pi x}{L}$

(e) $\boxed{u(x,t) = -2\sin\pi x\, e^{-10\pi^2 t} + 2\sin 4\pi x\, e^{-10(4\pi)^2 t}}$

(f) $\lim_{t\to\infty} u(x,t) = 0.$

④ (a) $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

$u(x,0) = 1 - \sin^2\frac{4\pi x}{L} + \cos\frac{\pi x}{L} = 1 - \left(\frac{\cos\frac{8\pi x}{L} - 1}{-2}\right) + \cos\frac{\pi x}{L}$

$= \frac{1}{2} + \cos\frac{\pi x}{L} + \frac{1}{2}\cos\frac{8\pi x}{L}$

Solution: $\boxed{u(x,t) = \frac{1}{2} + \cos\frac{\pi x}{L}\, e^{-k\left(\frac{\pi}{L}\right)^2 t} + \frac{1}{2}\cos\frac{8\pi x}{L}\, e^{-k\left(\frac{8\pi}{L}\right)^2 t}}$

(b) $\lim_{t\to\infty} u(x,t) = \frac{1}{2}$

(c) $E(t) = E(0) = c\rho A\int_0^L f(x)\,dx = c\rho A\int_0^L \left(\frac{1}{2} + \cos\frac{\pi x}{L} + \frac{1}{2}\cos\frac{8\pi x}{L}\right)dx = \frac{c\rho A L}{2}$

CONSERVED