NAME: ______________________

Practice Midterm Exam 2

1. (a) Solve $u_{tt} = u_{xx}$ for $x \in \mathbb{R}$, $t > 0$, with the initial data $u(x,0) = 2\sin x$ and $u_t(x,0) = 4\cos x$.

(b) What is the domain of dependence of the point $(x,t) = (3,1)$? Sketch it.

(c)What are the characteristic curves (or characteristic lines) passing through the point $(3,1)$ (write their equations)?

2. (a) What is the Sturm-Liouville eigenvalue problem that results from solving the following initial-boundary value problem for the heat equation with non-constant coefficients:

$$3xu_t = 5u_{xx}, x \in (0,L), T > 0,$$

and with the homogeneous Dirichlet boundary data $u(0,t) = 0$, $u(L,t) = 0$.

(b) What is the sign of the corresponding eigenvalues?
(*Recall:* Rayleigh quotient

$$\lambda = \frac{[-p\phi d\phi/dx]|_a^b + \int_a^B [p(d\phi/dx)^2 - q\phi^2]dx}{\int_a^b \phi^2 \sigma dx}, \text{ for } \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0.)$$

3. Consider a linearly elastic string on length L that is fixed at one end ($u(0,t) = 0$), while the other end of the string is moving with a prescribed motion $u(L,t) = \sin t$. Is the total energy of the string conserved? Prove your statement.

4. Find the solution of the Laplace's equation in 2D:

$$\triangle u = 0,\ x \in (0,1),\ y \in (0,2),$$

subject to the following boundary data:

$$\begin{aligned} &u(x,0) = 0, \ \ x \in (0,1) \\ &u(x,2) = 0,\ x \in (0,1) \\ &u(0,y) = \sin\frac{\pi y}{2},\ y \in (0,2) \\ &u(1,y) = 0,\ y \in (0,2). \end{aligned}$$

ANSWER KEY (EXAM 2 PRACTICE)

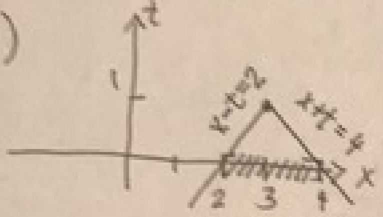
① (a) $u(x,t) = \frac{1}{2}\left[2\sin(x-t) + 2\sin(x+t)\right] + \frac{1}{\cancel{2}}\int_{x-t}^{x+t} \cancel{4}^{\,2} \cos\xi \, d\xi$

↑ D'Alambert formula

$= \sin(x-t) + \sin(x+t) + 2\sin(x+t) - 2\sin(x-t)$

$= -\sin(x-t) + 3\sin(x+t).$

(b)



$\mathcal{D}(3,1) = [2,4]$

(c) $x-t=2$ ← characteristic curves
$x+t=4$ ←

② (a) $u(x,t) = h(t)\phi(x) \Rightarrow 3x\, h'(t)\phi(x) = 5\, h(t)\phi''(x) \quad / \div h(t)\phi(x)\cdot 3x$

$$\frac{h'(t)}{h(t)} = \frac{5\,\phi''(x)}{3x\,\phi(x)} = -\lambda$$

Sturm-Liouville eigenvalue problem: $\boxed{\begin{array}{l} 5\phi''(x) + 3\lambda x \phi(x) = 0 \\ \phi(0) = 0 \\ \phi(L) = 0 \end{array}}$

(b) All eigenvalues are positive because the Rayleigh quotient implies

$$\lambda = \frac{\int_0^L 5\left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 \cdot 3x \, dx} \geq 0$$

But if $\lambda = 0$, that would imply $\phi \equiv 0$, which is not an eigenfunction. Thus all eigenvalues $\boxed{\lambda > 0}$ must be positive.

(3) $u_{tt} = c^2 u_{xx} \;/\; \cdot u_t \;/\; \int_0^L dx$

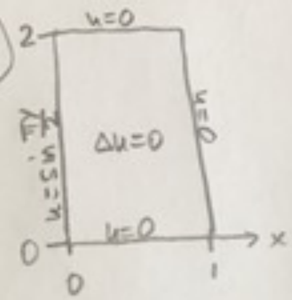
$$\int_0^L u_{tt}u_t\,dx = c^2\int_0^L u_{xx}u_t\,dx = -c^2\int_0^L u_x u_{tx}\,dx + \left[u_t u_x\right]_{x=0}^{L}$$

$$\Rightarrow \underbrace{\frac{1}{2}\frac{d}{dt}\int_0^L (u_t)^2\,dx + \frac{c^2}{2}\frac{d}{dt}\int_0^L (u_x)^2\,dx}_{\frac{dE(t)}{dt}} = c^2\left[u_t u_x\right]_{x=0}^{x=L}$$

$$\frac{dE(t)}{dt} = c^2\Big[\underbrace{u_t(L)}_{\cos t}u_x(L) - \underbrace{u_t(0)}_{=0}u_x(0)\Big] = c^2 u_x(L)\cos t \neq 0$$

$\Rightarrow$ $\boxed{\text{ENERGY IS } \underline{\text{NOT}} \text{ CONSERVED}}$

(4)



$u(x,y) = h(x)\phi(y)$

PDE $\Rightarrow$ $h''(x)\phi(y) = -h(x)\phi''(y) \Rightarrow \boxed{\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \lambda}$

ODE FOR $\phi(y)$: (EIGENVALUE PROBLEM)

$$\boxed{\begin{cases}\phi''(y) = -\lambda\phi(y),\ y\in(0,2)\\ \phi(0) = 0\\ \phi(2) = 0\end{cases}}$$

SOLUTION

$$\boxed{\begin{array}{l}\phi_n(y) = \sin\frac{n\pi y}{2}\\ \lambda_n = \left(\frac{n\pi}{2}\right)^2\\ n = 1,2,3\end{array}}$$

ODE FOR $h(x)$: $h''(x) - \lambda h(x) = 0$

Solution: $h(x) = C_1\cosh\sqrt{\lambda}(x-1) + C_2\sinh\sqrt{\lambda}(x-1)$

Bound. conditions: $h(1) = 0$, $h(0) = \sin\frac{\pi y}{2}$

From $h(1) = 0 \Rightarrow h(1) = C_1\cosh 0 + C_2\sinh 0 = C_1 = 0 \Rightarrow \boxed{C_1 = 0}$

Thus: $\boxed{h(x) = C_2\sinh\boxed{\sqrt{\lambda}(x-1)}}$. Since $\sqrt{\lambda} = \sqrt{\lambda_n} = \frac{n\pi}{2}$ we have

$$\boxed{h_n(x) = C_n\sinh\frac{n\pi}{2}(x-1)}$$

GENERAL SOLUTION: (*) $\boxed{u(x,y) = \sum_{n=1}^{\infty} C_n\sinh\frac{n\pi}{2}(x-1)\sin\frac{n\pi y}{2}}$

Obtain C_n from non-homogeneous data: $u(0,y) = \sin\frac{\pi y}{2}$.

From (*) $\Rightarrow u(0,y) = \sum_{n=1}^{\infty} C_n\sinh\frac{n\pi}{2}(-1)\sin\frac{n\pi y}{2} = \sin\frac{\pi y}{2}$

This implies:

$n=1$

$C_1\sinh\left(-\frac{\pi}{2}\right) = 1 \Rightarrow \boxed{C_1 = -\frac{1}{\sinh\frac{\pi}{2}}} \Rightarrow \boxed{u(x,y) = -\frac{1}{\sinh\frac{\pi}{2}}\sinh\frac{\pi}{2}(x-1)\sin\frac{\pi y}{2}}$

SOLUTION