NAME (please print):

Quiz 5 Sample

1. Consider the problem for a linearly elastic string fixed at the end points:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \ x \in (0, L), t > 0, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned}$$

Show the details in obtaining the solution of this initial-boundary value problem using the method of separation of variables.

2. Using what you found in problem 1, solve the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= 4u_{xx}, \ x \in (0, 1), t > 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= 0 \\ u_t(x,0) &= 3\sin 2\pi. \end{aligned}$$

3. Write and solve the ODEs that results from the initial boundary value problem for the damped string:

$$\begin{aligned} u_{tt} &= u_{xx} - u_t, \ x \in (0, 1), t > 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= f(x) \\ u_t(x,0) &= 0. \end{aligned}$$

Write the "general" solution as a superposition of separated solutions $u_n(x,t) = \phi_n(x) h_n(t)$ that you just found.

WE DID NOT COVER PROBLEMS 4 and 5 IN CLASS. WILL DO AFTER SPRING BREAK.

4. True or false: Every solution of the wave equation can be written as a sum of a forward moving traveling wave and a backward moving traveling wave.

5. Show that $u(x,t)$ given by the D'Alambert formula below satisfies the wave equation $u_{tt} = c^2 u_{xx}$:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Hint: Use $\frac{d}{dx}\int_{f_1(x)}^{f_2(x)} g(s)ds = f_2'(x)g(f_2(x)) - f_1'(x)g(f_1(x))$.
In fact, more generally, the following holds:

$$\frac{d}{dx}\int_{f_1(x)}^{f_2(x)} g(s,x)ds = f_2'(x)g(f_2(x)) - f_1'(x)g(f_1(x)) + \int_{f_1(x)}^{f_2(x)} \frac{\partial g}{\partial x}(s,x)ds.$$

Answer key

Problems 1-4 $\to$ in class.

(5) $u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

$u_t(x,t) = \frac{1}{2}\left[f'(x-ct)(-c) + f'(x+ct)(+c)\right] +$

$+ \frac{1}{2c}\left[c\,g(x+ct) - (-c)\,g(x-ct)\right]$

$$\boxed{\begin{aligned} u_{tt}(x,t) &= \frac{1}{2}\left[f''(x-ct)(-c)^2 + f''(x+ct)\,c^2\right] + \\ &+ \frac{1}{2c}\left[g'(x+ct)\cdot c^2 - c^2 g'(x-ct)\right] \end{aligned}} \quad (*)$$

$u_x(x,t) = \frac{1}{2}\left[f'(x-ct) + f'(x+ct)\right] + \frac{1}{2c}\left[g(x+ct) - g(x-ct)\right]$

$u_{xx}(x,t) = \frac{1}{2}\left[f''(x-ct) + f''(x+ct)\right] + \frac{1}{2c}\left[g'(x+ct) - g'(x-ct)\right]$

$$\boxed{c^2 u_{xx}(x,t) = \frac{c^2}{2}\left[f''(x-ct) + f''(x+ct)\right] + \frac{c}{2}\left[g'(x+ct) - g'(x-ct)\right]} \quad (**)$$

By comparing $(*)$ & $(**)$ we see that

$$u_{tt} = c^2 u_{xx}$$

for u given by D'Alembert Formula.