Existence and Uniqueness of a Solution to a Three-Dimensional Axially Symmetric Biot Problem Arising in Modeling Blood Flow

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Abstract. We prove the existence of a unique weak solution to a problem associated with studying blood flow in compliant, viscoelastic arteries. The model problem is a linearization of the leading-order approximation of a viscous, incompressible, Newtonian fluid flow in a long and slender viscoelastic tube with small aspect ratio. The resulting model is of Biot type. The linearized model equations form a hyperbolic-parabolic system of partial differential equations with degenerate diffusion. The degenerate diffusion is a consequence of the fact that the effects of the fluid viscosity in the axial direction of a long and slender tube are small in comparison with the effects of the fluid viscosity in the radial direction. Degenerate fluid diffusion and hyperbolicity of the hyperbolic-parabolic system cause lower regularity of a weak solution and are a source of the main difficulties associated with the existence proof. Crucial for the existence proof is the viscoelasticity of vessel walls which provides the main smoothing mechanisms in the energy estimates which, via the compactness arguments, leads to the proof of the existence of a solution of this problem. This has interesting consequences for the understanding of the underlying hemodynamics application. Our analysis shows that the viscoelasticity of the vessel walls is crucial in smoothing sharp wave fronts that might be generated by the steep pressure pulses emanating from the heart, which are known to occur in, for example, patients with aortic insufficiency.

1 Introduction

We study an initial-boundary value problem for the unknown functions γ and v_z

$$\gamma : (0,1) \times (0,T) \to \mathbb{R}, \quad \gamma : (z,t) \mapsto \gamma(z,t), \tag{1.1}$$

$$v_z : (0,1) \times (0,1) \times (0,T) \to \mathbb{R}, \quad v_z(r,z,t) \mapsto v_z(r,z,t), \tag{1.2}$$

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which satisfy

$$\frac{1}{\hat{\gamma}}\frac{\partial \gamma}{\partial t} + \frac{1}{\hat{\gamma}^2}\frac{\partial}{\partial z}\int_0^1 \hat{\gamma}^2 v_z \; r dr = 0, \tag{1.3}$$

$$\frac{\partial v_z}{\partial t} - \frac{C_1}{\hat{\gamma}^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) - \frac{1}{\hat{\gamma}}\frac{\partial \hat{\gamma}}{\partial t}\frac{\partial v_z}{\partial r} r = -C_2\frac{\partial \gamma}{\partial z} - C_3\frac{\partial^2 \gamma}{\partial z \partial t}, \tag{1.4}$$

in $\Omega \times (0,T)$ with $T > 0$ and

$$\Omega = \left\{x \in I\!R^3 : x = (r\cos\vartheta, r\sin\vartheta, z),\ 0 \le r < 1,\ \vartheta \in [0, 2\pi),\ 0 < z < 1\right\}. \tag{1.5}$$

Function $\hat{\gamma}$ in (1.3), (1.4) is given and is such that $\hat{\gamma} = \hat{\gamma}(z,t) \ge \delta > 0$ for some $\delta > 0$ and $\forall(z,t) \in (0,1)\times(0,T)$, and C_1, C_2 and C_3 are positive constants, i.e., $C_1, C_2, C_3 > 0$.

The initial and boundary data are given by:

$$\begin{cases} \gamma(0,t) = \gamma_0(t),\ \gamma(1,t) = \gamma_1(t),\ \ \gamma(z,0) = \gamma^0(z), \\ v_z(r=1,z,t) = 0,\ \ v_z(r,z,t=0) = v_z^0(r,z),\ \ |v_z(r=0,z,t)| < +\infty. \end{cases} \tag{1.6}$$

This initial-boundary value problem is motivated by a study of blood flow in pulsatile arteries [10, 11]. Versions of this model also appear in the studies of viscous, incompressible flow through elastic porous media with elastic structure undergoing small vibrations (the Biot model) [3, 5, 6, 15, 17]. In the blood flow application the model is derived by considering medium-to-large arteries where blood can be modeled as an incompressible, viscous fluid, utilizing the incompressible Navier-Stokes equations to model the flow. The Navier-Stokes equations are coupled to the equations for a viscoelastic membrane (Kelvin-Voigt viscoelasticity) modeling the mechanical behavior of arterial walls. See [2, 10, 11, 16]. The resulting coupled problem is a nonlinear moving-boundary problem defined on a cylindrical domain with reference radius R and reference length L corresponding to a section of a blood vessel. Assuming small aspect ratio $\epsilon = R/L$ of the cylindrical domain and axially symmetric flow, a set of closed, reduced, effective equations in cylindrical coordinates was first derived in [8] for the Stokes problem and linearly elastic structure. This was extended in [11] to the incompressible Navier-Stokes equations coupled with a linearly elastic structure, and then in [10] to the Navier-Stokes equations coupled with the equations of a linearly viscoelastic structure (membrane and Koiter shell with Kelvin-Voigt viscoelasticity). It was proved in [8, 11] that this reduced, effective problem approximates the original problem to the ϵ^2 accuracy. In contrasts with the "classical" one-dimensional models, see e.g. [4, 7], the models derived in [8, 10, 11] do not require any *ad hoc* closure assumptions on the form of the velocity profile, since they were derived in a consistent way giving rise to a *closed* problem in which the velocity profile follows from the solution of the problem itself. The models derived in [8, 10, 11]

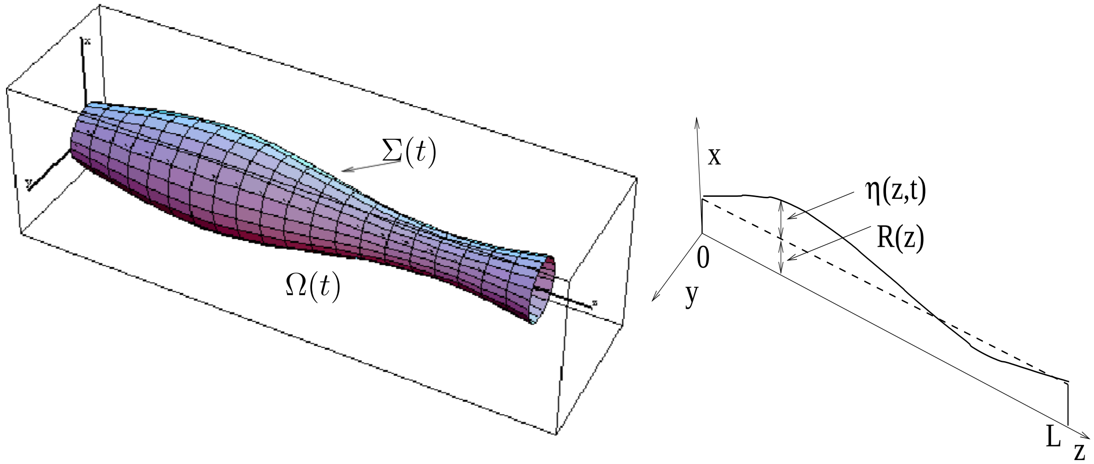

Figure 1.1: The moving domain $\Omega(t)$. Domain radius $\gamma(z,t) = R(z) + \eta(z,t)$.

capture the leading order physics of the flow of a viscous, incompressible Newtonian fluid in a cylindrical tube with elastic/viscoelastic membrane/Koiter shell walls.

The leading-order *effective equations* derived in [10] are in the form of a nonlinear, moving-boundary problem for a system of partial differential equations of mixed hyperbolic-parabolic type. They are given in terms of the unknown functions v_z and γ where:

- v_z is the axial component of the fluid velocity, and
- γ is the radius of the tube wall.

The (leading-order) nonlinear moving-boundary problem holds in the cylindrical domain $\Omega(t)$

$$\Omega(t) = \left\{x \in I\!R^3 : x = (r\cos\vartheta, r\sin\vartheta, z),\ 0 \le r < \gamma(z,t),\ \vartheta \in [0,2\pi),\ 0 < z < L\right\}, \tag{1.7}$$

for $0 < t < T$, with $T > 0$. See Figure 1.1. The reference configuration corresponds to that of a straight cylinder with radius R and length L. The effective reduced problem, written in terms of the vessel wall radius $\gamma(z,t)$, the axial component of the fluid velocity $v_z(r,z,t)$, and the fluid pressure $p(z,t)$, reads [10, 11]:

$$\frac{\partial \gamma^2}{\partial t} + \frac{\partial}{\partial z}\int_0^\gamma 2 r v_z dr = 0, \quad 0 < z < L,\ 0 < t < T, \tag{1.8}$$

$$\varrho_F \frac{\partial v_z}{\partial t} - \mu_F \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{\partial p}{\partial z}, \quad (z,r,t) \in \Omega(t) \times (0,T), \tag{1.9}$$

with

$$p - p_{\text{ref}} = \left(\frac{hE}{R(1-\sigma^2)} + p_{\text{ref}}\right)\left(\frac{\gamma}{R} - 1\right) + \frac{hC_v}{R^2}\frac{\partial \gamma}{\partial t}, \quad 0 < z < L,\ 0 < t < T. \tag{1.10}$$

It is obvious that the pressure can be eliminated from the problem, giving rise to a 2×2 system of partial differential equations written in terms of γ and v_z.

Here ϱ_F is the fluid density, μ_F is the fluid dynamic viscosity coefficient, p is the fluid pressure with p_{ref} denoting the pressure at which the reference configuration is assumed. The constants describing the structure properties are the Young's modulus of elasticity E, the Poisson ratio σ, the wall thickness h, and the structure viscoelasticity constant C_v. The first equation (1.8), derived from the conservation of mass, describes the transport of $(\gamma)^2$ with the averaged fluid velocity $U := \frac{1}{(\gamma)^2}\int_0^\gamma v_z r dr$, while the second equation (1.9), derived from the balance of momentum, incorporates diffusion due to the fluid viscosity which is dominant in the radial direction. The diffusion in the axial direction z is of order ϵ^2 and thus drops out from the leading-order effective equations [10, 11]. Furthermore, the nonlinear fluid advection term turns out to be of order ϵ, thereby appearing only in the ϵ-correction of the leading-order equations which are not shown here since they are easy to calculate [10]. Equation (1.10) is the leading-order approximation of the coupling between the fluid contact force and the contact force of the vessel wall which is modeled as a linearly viscoelastic membrane.

Problem (1.8)-(1.10) is supplemented by the following initial and boundary conditions describing pressure-driven flow in a compliant cylinder:

$$v_z(0,z,t) - \text{ bounded}, \quad v_z(\gamma(z,t),z,t) = 0, \quad v_z(r,z,0) = v_z^0, \tag{1.11}$$

$$\gamma(z,0) = \gamma^0, \quad p(0,t) = P_0(t), \quad p(L,t) = P_L(t). \tag{1.12}$$

Problem (1.3)-(1.4) studied in this manuscript, is obtained after the nonlinear moving-boundary problem (1.8)-(1.12) is mapped onto a *fixed, scaled* domain and the resulting problem is linearized around a given function $\gamma = \hat{\gamma}$ and $v_z = 0$. In terms of the physical parameters in the problem, constants C_1, C_2 and C_3 are given by

$$C_1 = \frac{\mu_F \tau}{\rho_F R^2},\ C_2 = \left(\frac{Eh}{(1-\sigma^2)R}K + p_{ref}\right)\frac{1}{V^2 \varrho_F},\ C_3 = \frac{hC_v}{RLV\varrho_F}, \tag{1.13}$$

where τ is the time scale and V determines the scale for the velocity. The existence of a unique solution to this linearized problem is a building block in the existence proof of the corresponding nonlinear problem. Results and estimates presented in this manuscript will be crucial in the proof of the existence of a nonlinear solution which is a perturbation of the zero-velocity flow in a vessel with constant radius, and, even more interestingly, a perturbation of the Womersley flow in a tube with constant radius allowing relatively large pressure gradient and relatively large steepness of the pressure pulse. See [14].

The linear problem (1.3)-(1.6) itself, however, is non-trivial and is interesting in its own right. The main reasons for this are the following: the system combines a **hyperbolic with a parabolic** equation, with **degenerate fluid diffusion** in the balance of momentum (parabolic) equation (1.4). Degenerate fluid diffusion in z-direction implies no boundary conditions for v_z at the inlet and outlet boundaries. Hyperbolicity of the hyperbolic-parabolic system and degenerate diffusion give rise to a weak solution which is less regular than the standard parabolic solutions [12], and it requires special tricks to prove its existence. In particular, it was crucial for us to notice that the cross-sectional average of the fluid velocity plays a special role in this problem. Namely, although the z-derivative of the velocity itself is not in L^2 for a weak solution, the z-derivative of its average *is* in L^2. This was included in the definition of the solution space for the velocity and it provided a necessary ingredient that allowed us to prove the existence result.

All the smoothing in this problem comes from the second equation (1.4), as the first equation (1.3) is just a transport problem. It was crucial in the proof of the existence of a weak solution that the structure viscoelasticity was not zero, namely, that the coefficient $C_3 \neq 0$. This enabled an energy estimate which shows regularization of the (time) evolution of γ, which, in turn, provides square integrability of the z-derivative of the average fluid velocity via the transport problem (1.3).

This has interesting consequences on the blood flow application problem. Our analysis shows that the viscoelasticity of vessel walls is crucial in smoothing out the potentially sharp wave fronts in the velocity and wall displacement which might be generated by the steep pressure pulse emanating from the heart, as is the case, for example, in patients with aortic insufficiency [7, 13].

This manuscript is organized as follows. In Section 2 we introduce an equivalent formulation of the problem in which the unknown function γ satisfies homogeneous boundary data. In Section 3 we introduce the solution spaces and define weak solution, and in Section 4 we prove the existence of a unique weak solution by using Galerkin approximations, energy estimates and compactness arguments. Finally, in Section 5 we show higher regularity of our weak solution which will imply that system (1.3)-(1.4) is satisfied by (γ, v_z) for a.a. $(r, z, t) \in \Omega \times (0, T)$.

2 The Problem with Homogeneous Inlet and Outlet Boundary Data

We first rewrite problem (1.3)-(1.6) in terms of the function $\bar{\gamma}$ which satisfies homogeneous boundary data.

Assuming that the inlet and outlet boundary data γ_0 and γ_1 are smooth enough we can introduce the function $\bar{\gamma}$

$$\bar{\gamma}(z,t) = \gamma(z,t) - ((\gamma_1(t) - \gamma_0(t))z + \gamma_0(t)) \tag{2.1}$$

that vanishes at the inlet and outlet boundary $z = 0$ and $z = 1$. Our problem can then be re-written in terms of $\bar{\gamma}$ as follows:

$$\frac{1}{\hat{\gamma}}\frac{\partial \bar{\gamma}}{\partial t} + \frac{1}{\hat{\gamma}^2}\frac{\partial}{\partial z}\int_0^1 \hat{\gamma}^2 v_z \; r dr + \frac{1}{\hat{\gamma}}F_1 = 0, \tag{2.2}$$

$$\frac{\partial v_z}{\partial t} - \frac{C_1}{\hat{\gamma}^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) - \frac{1}{\hat{\gamma}}\frac{\partial \hat{\gamma}}{\partial t}\frac{\partial v_z}{\partial r}r = -C_2\frac{\partial \bar{\gamma}}{\partial z} - C_3\frac{\partial^2 \bar{\gamma}}{\partial z \partial t} - F_2. \tag{2.3}$$

where $\bar{\gamma}$ and v_z satisfy the following initial and boundary data:

$$\begin{cases} \bar{\gamma}(z,0) = \gamma^0(z) - ((\gamma_1(0) - \gamma_0(0))z - \gamma_0(0)) =: \bar{\gamma}^0(z), \; \bar{\gamma}(1,t) = \bar{\gamma}(0,t) = 0, \\ v_z(r=1,z,t) = 0, \; v_z(t=0,z,t) = 0, \; |v_z(r=0,z,t)| < +\infty. \end{cases} \tag{2.4}$$

In equations (2.2) and (2.3) functions F_1 and F_2 are given by:

$$F_1(z,t) = (\gamma_1'(t) - \gamma_0'(t))z + \gamma_0'(t), \tag{2.5}$$

$$F_2(t) = C_2(\gamma_1(t) - \gamma_0(t)) + C_3(\gamma_L'(t) - \gamma_0'(t)). \tag{2.6}$$

3 Solution Spaces and Definition of a Weak Solution

We are interested in the solution to problem (1.3)-(1.6) defined for a fixed, given function $\hat{\gamma}$ (a "linearization" of the scaled radius γ) which is bounded away from zero, and its maximum value is strictly less than two times the radius of the reference domain $r = 1$. For simplicity, take the bound on the maximum radius to be $3/2$. More precisely, we require that $\hat{\gamma}$ belongs to a set $\hat{\Gamma}$:

$$\hat{\Gamma} = \left\{ \hat{\gamma} \in H^1(0,T : C^1[0,1]) \mid \min_{z,t} \hat{\gamma}(z,t) \geq \delta > 0, \; \max_{0 \leq t \leq T} \|\hat{\gamma}(\cdot,t)\|_{C[0,1]} \leq \frac{3}{2} \right\}, \tag{3.1}$$

for some fixed time $T > 0$.

Here $H^1(0,T : C^1[0,1])$ consists of all the functions $f \in L^2(0,T : C^1[0,1])$ such that $\partial f/\partial t$ exists in the weak sense and belongs to $L^2(0,T : C^1[0,1])$. The norm is given by

$$\|f\|^2_{H^1(0,T:C^1[0,1])} = \int_0^T \left(\|f\|^2_{C^1[0,1]} + \left\| \frac{\partial f}{\partial t} \right\|^2_{C^1[0,1]} \right) \, dt.$$

The space $C^1[0,1]$ consists of all the bounded and uniformly continuous functions g such that g' is bounded and uniformly continuous on $(0,1)$. $C^1[0,1]$ is a Banach space with the norm

$$\|g\|_{C^1[0,1]} := \max_{0 \leq |j| \leq 1} \sup_{z \in (0,1)} \left| g^j(z) \right|.$$

For a given $\hat{\gamma} \in \hat{\Gamma}$ and smooth boundary data, we will be looking for a weak solution $(\bar{\gamma}, v_z) \in \Gamma \times V$ to problem (2.2)-(2.6) where the solution spaces Γ and V are defined as follows:

$$\Gamma = H^1(0, T : L^2(0,1)), \tag{3.2}$$

and the axial velocity v_z will be defined on the Sobolev space with a weighted norm associated with the axial symmetry of the problem. To write a definition of the solution space for v_z we introduce the following notation. We will say that $u \in L^2(\Omega, r)$ if the weighted L^2 – norm with the weight r is bounded, i.e.,

$$\int_\Omega |u|^2 \, r dr dz < +\infty.$$

Define $H^1_{0,0}(\Omega, r)$ by the following

$$H^1_{0,0}(\Omega, r) = \Big\{ w \in L^2(\Omega, r) : \frac{\partial w}{\partial r} \in L^2(\Omega, r), \ \int_0^1 w \ r dr \in H^1(0,1), \\ w|_{r=1} = 0, \ | \ w|_{r=0} \ | < +\infty \Big\}.$$

The solution space for the axial component of velocity is defined as follows

$$V = \{ w \in L^2(0, T : H^1_{0,0}(\Omega, r)) : \frac{\partial w}{\partial t} \in L^2(0, T : H^{-1}_{0,0}(\Omega, r)) \}. \tag{3.3}$$

The norm on $H^1_{0,0}(\Omega, r)$ is given by:

$$\|w\|^2_{H^1_{0,0}(\Omega,r)} = \int_\Omega \left(|w|^2 + \left| \frac{\partial w}{\partial r} \right|^2 \right) \ r dr dz + \int_0^1 \left| \frac{\partial}{\partial z} \int_0^1 w \ r dr \right|^2 dz.$$

Definition 3.1 *We say that $(\bar{\gamma}, v_z) \in \Gamma \times V$ is a weak solution to the linear problem (2.2)-(2.6) provided that for all $\psi \in H^1_0(0,1)$ and $w \in H^1_{0,0}(\Omega, r)$ the following holds*

$$\int_0^1 \frac{1}{\hat{\gamma}} \frac{\partial \bar{\gamma}}{\partial t} \psi \ dz + \int_0^1 \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} \psi \int_0^1 v_z r dr dz - \int_0^1 \frac{\partial \psi}{\partial z} \int_0^1 v_z r dr dz + \int_0^1 \frac{1}{\hat{\gamma}} F_1 \psi \ dz = 0 \tag{3.4}$$

$$\int_\Omega \frac{\partial v_z}{\partial t} w \ r dr dz + C_1 \int_\Omega \frac{1}{\hat{\gamma}^2} \frac{\partial v_z}{\partial r} \frac{\partial w}{\partial r} \ r dr dz - \int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} \frac{\partial v_z}{\partial r} w \ r^2 dr dz \\ = C_2 \int_0^1 \bar{\gamma} \frac{\partial}{\partial z} \int_0^1 w \ r dr \ dz + C_3 \int_0^1 \frac{\partial \bar{\gamma}}{\partial t} \frac{\partial}{\partial z} \int_0^1 w \ r dr \ dz - \int_\Omega F_2 w \ r dr dz \tag{3.5}$$

with

$$\bar{\gamma}(z, 0) = \bar{\gamma}^0(z), \ v_z(r, z, t = 0) = 0. \tag{3.6}$$

4 Existence of a Unique Weak Solution

In this section we prove that there exists a unique weak solution to problem (2.2)-(2.6).

Theorem 4.1 *Assume that the initial data $v_z^0(r,z)$ and $\bar{\gamma}^0(z)$ satisfy $v_z^0 \in L^2(\Omega, r)$ and $\bar{\gamma}^0 \in L^2(0,1)$, and that the boundary data $\gamma_0(t)$ and $\gamma_1(t)$ satisfy $\gamma_0, \gamma_1 \in H^1(0,T)$. Then, for each given $\hat{\gamma} \in \hat{\Gamma}$ there exists a unique weak solution $(\bar{\gamma}, v_z) \in \Gamma \times V$ of problem (2.2) - (2.4).*

The proof is an application of the Galerkin method involving non-trivial energy estimates. Additional difficulty is imposed by the singular weight at $r=0$ and by the fact that the coefficients of the problem depend on both z and t. We handle these difficulties and obtain the proof by performing the following three classical steps:

1. Construction of a finite-dimensional approximation of the solution by the Galerkin method.
2. Energy estimates that provide a uniform bound for the sequence of Galerkin approximations.
3. Passing to the limit in the weighted norm using compactness arguments.

4.1 Galerkin Approximation

Let $\{\phi_k\}_{k=1}^{\infty}$ be the smooth functions which are orthogonal in $H_0^1(0,1)$, orthonormal in $L^2(0,1)$ and span the solution space for $\bar{\gamma}$. Additionally, the functions ϕ_i are chosen to be the eigenfunctions for $-\Delta$ on $H_0^1(0,1)$, that is, $-\Delta\phi_i = \lambda_i \phi_i$. Furthermore, introduce the smooth functions $\{w_k\}_{k=1}^{\infty}$ which satisfy $w_k|_{r=1} = 0$, and are orthonormal in $L^2(\Omega, r)$ and span the solution space for the velocity v_z. Introduce the function space

$$C_{0,0}^k(\Omega) = \left\{ v \in C^k(\Omega) : v|_{r=1} = 0 \right\},$$

for any $k = 0, 1, ..., \infty$.

Fix positive integers m and n. We look for the functions $\bar{\gamma}_m : [0,T] \to C_0^{\infty}(0,1)$ and $v_{z_n} : [0,T] \to C_{0,0}^{\infty}(\Omega)$ of the form

$$\bar{\gamma}_m(t) = \sum_{i=1}^{m} d_i^m(t)\, \phi_i, \tag{4.1}$$

$$v_{z_n}(t) = \sum_{j=1}^{n} l_j^n(t)\, w_j, \tag{4.2}$$

where the coefficient functions d_h^m and l_k^n are chosen so that the functions $\bar\gamma_m$ and v_{z_n} satisfy the weak formulation (3.4)-(3.5) of the linear problem (2.2), (2.4), projected onto the finite dimensional subspaces spanned by $\{\phi_i\}$ and $\{w_j\}$ respectively:

$$\int_0^1 \frac{1}{\hat\gamma}\frac{\partial\bar\gamma_m}{\partial t}\phi_h dz + \int_0^1 \frac{2}{\hat\gamma}\frac{\partial\hat\gamma}{\partial z}\phi_h \int_0^1 v_{z_n} r dr dz - \int_0^1 \frac{\partial\phi_h}{\partial z}\int_0^1 v_{z_n} r dr dz + \int_0^1 \frac{1}{\hat\gamma}F_1\phi_h dz = 0, \tag{4.3}$$

$$\begin{aligned}&\int_\Omega \frac{\partial v_{z_n}}{\partial t} w_k\ r dr dz + C_1\int_\Omega \frac{1}{\hat\gamma^2}\frac{\partial v_{z_n}}{\partial r}\frac{\partial w_k}{\partial r}\ r dr dz - \int_\Omega \frac{1}{\hat\gamma}\frac{\partial\hat\gamma}{\partial t}\frac{\partial v_{z_n}}{\partial r} w_k\ r^2 dr dz\\ &= C_2\int_0^1 \bar\gamma_m \frac{\partial}{\partial z}\int_0^1 w_k\ r dr\ dz + C_3\int_0^1 \frac{\partial\bar\gamma_m}{\partial t}\frac{\partial}{\partial z}\int_0^1 w_k\ r dr\ dz - \int_\Omega F_2 w_k\ r dr dz\end{aligned} \tag{4.4}$$

for $a.e.$ $0\le t\le T,\ h=1,\cdots,m$, and $k=1,\cdots,n$, and with the initial data

$$\begin{cases} d_h^m(0) = (\bar\gamma^0(z),\phi_h(z))_{L^2(0,1)},\\ l_k^n(0) = (v_z^0(r,z), w_k(r,z))_{L^2(\Omega,r)}.\end{cases} \tag{4.5}$$

Here $(\ ,\ )_{L^2(0,1)}$ and $(\ ,\ )_{L^2(\Omega,r)}$ denote the inner product in $L^2(0,1)$ and $L^2(\Omega,r)$ respectively. The existence of the coefficient functions satisfying these requirements is guaranteed by the following Lemma.

LEMMA 4.2 *For each $m=1,2,...$ and $n=1,2,...$ there exist unique functions $\bar\gamma_m$ and v_{z_n} of the form (4.1) and (4.2), respectively, satisfying* (4.3)-(4.5). *Moreover*

$$(\bar\gamma_m, v_{z_n}) \in H^1(0,T:C_0^\infty(0,1))\times H^1(0,T:C_{0,0}^\infty(\Omega)).$$

PROOF: To simplify notation, let us first introduce the following vector functions

$$d^m(t) = \begin{pmatrix} d_1^m(t)\\ \vdots\\ d_m^m(t)\end{pmatrix},\quad l^n(t) = \begin{pmatrix} l_1^n(t)\\ \vdots\\ l_n^n(t)\end{pmatrix},\quad Y(t) = \begin{pmatrix} d^m(t)\\ l^n(t))\end{pmatrix}. \tag{4.6}$$

Then, equation (4.3) written in matrix form reads:

$$A_1(t)d^{m'}(t) + A_2(t)l^n(t) + S_1(t) = 0, \tag{4.7}$$

where A_1 is an $m\times m$ matrix, A_2 an $m\times n$ matrix and S_1 an $m\times 1$ matrix defined by the following:

$$\begin{aligned}[A_1(t)]_{h,i} &= \left(\frac{1}{\hat\gamma}\phi_i,\phi_h\right)_{L^2(0,1)},\\ [A_2(t)]_{h,i} &= \left(\frac{2}{\hat\gamma}\frac{\partial\hat\gamma}{\partial z}\int_0^1 w_j r dr,\phi_h\right)_{L^2(0,1)} - \left(\int_0^1 w_j r dr, \frac{\partial\phi_h}{\partial z}\right)_{L^2(0,1)}\\ [S_1(t)]_{h,1} &= \left(\frac{1}{\hat\gamma}F_1,\phi_h\right)_{L^2(0,1)}\end{aligned}$$

where $h, i = 1, ..., m$ and $j = 1, ..., n$. Similarly, equation (4.4) written in matrix form reads:

$$B_1(t) l^{n'}(t) + B_2(t) l^n(t) = B_3(t) d^m(t) + B_4(t) d^{m'}(t) - S_2(t), \tag{4.8}$$

where $B_1(t)$ and $B_2(t)$ are $n \times n$ matrices, $B_3(t)$ and $B_4(t)$ are $n \times m$ matrices, and $S_2(t)$ is an $n \times 1$ matrix defined by the following:

$$\begin{aligned}
[B_1(t)]_{k,j} &= (w_j, w_k)_{L^2(\Omega,r)} = \delta_{k,j} \\
[B_2(t)]_{k,j} &= C_1 \left(\frac{1}{\hat{\gamma}^2} \frac{\partial w_j}{\partial r}, \frac{\partial w_k}{\partial r} \right)_{L^2(\Omega,r)} - \left(\frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} \frac{\partial w_j}{\partial r} r, w_k \right)_{L^2(\Omega,r)}, \\
[B_3(t)]_{k,i} &= C_2 \left(\frac{\partial}{\partial z} \int_0^1 w_k \; r dr, \phi_i \right)_{L^2(0,1)}, \\
[B_4(t)]_{k,i} &= C_3 \left(\frac{\partial}{\partial z} \int_0^1 w_k \; r dr, \phi_i \right)_{L^2(0,1)}, \\
[S_2(t)]_{k,1} &= (F_2, w_k)_{L^2(\Omega,r)},
\end{aligned}$$

where $k, j = 1, ..., n$ and $i = 1, ..., m$.

Equations (4.7) and (4.8) can be written together as the following system

$$\begin{cases} A(t) Y'(t) + B(t) Y(t) = S(t), \\ Y(0) = \begin{pmatrix} d^m(0) \\ l_k^n(0) \end{pmatrix}, \end{cases} \tag{4.9}$$

where Y is defined in (4.6) and

$$\begin{aligned}
A(t) &= \begin{pmatrix} A_1^{m\times m}(t) & 0^{m\times n} \\ -B_4^{n\times m}(t) & B_1^{n\times n}(t) \end{pmatrix}_{(m+n)\times(m+n)}, \\
B(t) &= \begin{pmatrix} 0^{m\times m} & A_2^{m\times n}(t) \\ -B_3^{n\times m}(t) & B_2^{n\times n}(t) \end{pmatrix}_{(m+n)\times(m+n)}, \\
S(t) &= \begin{pmatrix} -S_1(t) \\ -S_2(t) \end{pmatrix}_{(m+n)\times 1}.
\end{aligned}$$

Function S incorporates the initial and boundary data obtained from the right hand-sides of equations (4.7) and (4.8).

To guarantee the existence of a solution $Y(t)$ of appropriate regularity first notice that linear independence of the sets $\{\phi_1, \cdots, \phi_m\}$ and $\{w_1, \cdots, w_n\}$, and uniform boundedness of $\hat{\gamma}(z,t)$ away from zero guarantee that the matrix $A(t)$ is nonsingular for all $t \in [0,T]$. Additionally, since $\hat{\gamma} \in \hat{\Gamma}$, the coefficient matrix functions A_1 and A_2 are in $L^\infty(0,T)$, and the coefficient matrix function B_2 is in $L^2(0,T)$. This is sufficient to guarantee the existence of an $[H^1(0,T)]^{m+n}$

function $Y(t) = (d^m(t), l^n(t))$ satisfying (4.9) for a.e $0 \le t \le T$. Therefore, functions $(\bar{\gamma}_m, v_{z_n})$, defined via $d^m(t)$ and $l^n(t)$ in (4.1), solve (4.3), (4.4) and (4.5) for a.e. $0 \le t \le T$. Moreover,

$$(\bar{\gamma}_m, v_{z_n}) \in H^1(0, T : C_0^\infty(0,1)) \times H^1(0, T : C_{0,0}^\infty(\Omega)).$$

■

4.2 Energy Estimates

In this section we derive an energy estimate for $\bar{\gamma}_m$ and v_{z_n} which is uniform in m and n. The estimate will bound the L^2-norms of $\bar{\gamma}_m$ and v_{z_n}, the L^2-norms of $\frac{\partial v_{z_n}}{\partial r}$ and $\frac{\partial \bar{\gamma}_m}{\partial t}$, and the $L^2(0, T : H_{0,0}^{-1}(\Omega, r))$-norm of $\frac{\partial v_{z_n}}{\partial t}$, in terms of the initial and boundary data and the coefficients of (2.2)-(2.6).

Theorem 4.3 **(Energy Estimate)** *There exists a constant C, depending only on Ω, T, and the coefficients of (2.2)-(2.6) such that*

$$\begin{aligned}
&\max_{0\le t\le T}\left\{\|v_{z_n}\|^2_{L^2(\Omega,r)} + \frac{C_2}{\hat{\gamma}_{\max}}\|\bar{\gamma}_m\|^2_{L^2(0,1)}\right\} \\
&\quad + \frac{C_1}{\hat{\gamma}^2_{\max}}\left\|\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(0,T:L^2(\Omega,r))} + \frac{C_3}{\hat{\gamma}_{\max}}\left\|\frac{\partial \bar{\gamma}_m}{\partial t}\right\|^2_{L^2(0,T:L^2(0,1))} + \left\|\frac{\partial v_{z_n}}{\partial t}\right\|_{L^2(0,T:H^{-1}_{0,0}(\Omega,r))} \\
&\quad \le C\left(\|v_z^0\|^2_{L^2(\Omega,r)} + \|\bar{\gamma}^0\|^2_{L^2(0,1)} + \|\gamma_1 - \gamma_0\|^2_{H^1(0,T)} + \|\gamma_0'\|^2_{L^2(0,T)}\right) \qquad (4.10)
\end{aligned}$$

where $\hat{\gamma}_{\max} = \max_{0\le t\le T}\|\hat{\gamma}(\cdot,t)\|_{C[0,1]}$, v_z^0 and $\bar{\gamma}^0$ are the initial data for v_z and $\bar{\gamma}$ respectively, and γ_0 and γ_1, which comprise the source terms in (2.2)-(2.6), are the boundary data (inlet/outlet) for γ. Constant C depends on the coefficients of (2.2)-(2.6) via C_1, C_2, C_3, $\delta = \min_{z,t}\hat{\gamma}$, and $\|\hat{\gamma}\|_{H^1(0,T:C[0,1])}, \|\hat{\gamma}\|_{L^2(0,T:C^1[0,1])}$.

Furthermore,

$$\frac{\partial}{\partial z}\int_0^1 v_{z_n} r dr \in L^2(0, T : L^2(0,1)),$$

and its $L^2(0, T : L^2(0,1))$-norm is bounded by the right hand-side of the energy estimate (4.10)*.*

Notice that this energy estimate is given in terms of the boundary data for γ, which, via (1.10), is related to the pressure data for the original problem. Thus, energy estimate (4.10) is an estimate given in terms of the L^2 norm of the pressure gradient in the tube via the term $\|\gamma_1 - \gamma_0\|^2_{H^1(0,T)}$ and the L^2-norm of the inlet pressure via the term $\|\gamma_0'\|^2_{L^2(0,T)}$.

Proof: We aim at using the Gronwall's inequality to obtain the final estimate. However, the lack of smoothness in z gives rise to problems related to controlling the z-derivative of $\bar{\gamma}_m$, which appears in the estimate if a standard approach to the energy estimate were taken. In order to deal with this problem, we will manipulate the conservation of mass and balance of momentum equations in order to cancel the unwanted terms that include the derivative of $\bar{\gamma}_m$ with respect z for which we have no control at this point. Similar difficulties arise due to the lack of smoothness in z of v_z. We again manipulate the equations in order to cancel the z derivatives of the cross-sectional average of v_z. However, we will be able to show in the end that, in fact, the z-derivative of the average of the velocity is in $L^2(0,1)$, as required by the definition of a weak solution. It is this lack of smoothness in the z direction and the presence of non-constant coefficients given in terms of the function $\hat{\gamma}$, that makes this proof challenging.

For simplicity, we will be using $\|\cdot\|_C$ to denote the $C[0,1]$-norm of a function of z.

STEP 1. We first manipulate the conservation of mass equation and balance of momentum in order to cancel out the z-derivatives of $\bar{\gamma}_m$ and of $\int_0^1 v_{z_n} r dr$ and obtain an estimate which will be used as a base for Gronwall's inequality.

Multiply (4.4) by $l_k^n(t)$ and sum over k for $k = 1, \cdots, n$ to find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |v_{z_n}|^2 \ r dr dz + C_1 \int_\Omega \frac{1}{\hat{\gamma}^2}\left|\frac{\partial v_{z_n}}{\partial r}\right|^2 \ r dr dz \\
&- \int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial \hat{\gamma}}{\partial t}\frac{\partial v_{z_n}}{\partial r} v_{z_n} \ r^2 dr dz = \underbrace{C_2 \int_0^1 \bar{\gamma}_m \frac{\partial}{\partial z}\int_0^1 v_{z_n} \ r dr \ dz}_{(i)} \\
&+ \underbrace{C_3 \int_0^1 \frac{\partial \bar{\gamma}_m}{\partial t}\frac{\partial}{\partial z}\int_0^1 v_{z_n} \ r dr \ dz}_{(ii)} - \int_\Omega F_2 v_{z_n} \ r dr dz
\end{aligned} \tag{4.11}$$

Next, multiply (4.3) by $C_2 d_h^m(t)$ and sum over h for $h = 1, \cdots, m$ to find

$$\begin{aligned}
&\frac{C_2}{2}\frac{d}{dt}\int_0^1 \frac{1}{\hat{\gamma}}|\bar{\gamma}_m|^2 \ dz + \frac{C_2}{2}\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial \hat{\gamma}}{\partial t}|\bar{\gamma}_m|^2 \ dz + C_2 \int_\Omega \frac{2}{\hat{\gamma}}\frac{\partial \hat{\gamma}}{\partial z} v_{z_n}\bar{\gamma}_m \ r dr dz \\
&\underbrace{- C_2 \int_0^1 \int_0^1 v_{z_n} \ r dr \ \frac{\partial \bar{\gamma}_m}{\partial z} \ dz}_{(i)} + C_2 \int_0^1 \frac{1}{\hat{\gamma}} F_1 \bar{\gamma}_m \ dz = 0,
\end{aligned} \tag{4.12}$$

Furthermore, multiply (4.3) by $C_3\dot{d}_h^m(t)$ and sum over h for $h=1,\cdots,m$ to find

$$C_3\int_0^1 \frac{1}{\hat{\gamma}}\left|\frac{\partial\bar{\gamma}_m}{\partial t}\right|^2\ dz + C_3\int_\Omega \frac{2}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}v_{z_n}\frac{\partial\bar{\gamma}_m}{\partial t}\ rdrdz$$
$$\underbrace{-\,C_3\int_0^1\int_0^1 v_{z_n}\ rdr\ \frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ dz}_{(ii)}+C_3\int_0^1 \frac{1}{\hat{\gamma}}F_1\frac{\partial\bar{\gamma}_m}{\partial t}\ dz=0, \tag{4.13}$$

Add the three equations (4.11)–(4.13). After the cancellation of the terms denoted by (i) and (ii) we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\|v_{z_n}\|^2_{L^2(\Omega,r)}+C_2\left\|\frac{1}{\sqrt{\hat{\gamma}}}\bar{\gamma}_m\right\|^2_{L^2(0,1)}\right\}+C_1\left\|\frac{1}{\hat{\gamma}}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)}+C_3\left\|\frac{1}{\sqrt{\hat{\gamma}}}\frac{\partial\bar{\gamma}_m}{\partial t}\right\|^2_{L^2(0,1)}$$
$$=\int_\Omega\frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial v_{z_n}}{\partial r}v_{z_n}\ r^2drdz-\frac{C_2}{2}\int_0^1\frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}\left|\bar{\gamma}_m\right|^2\ dz-C_2\int_\Omega\frac{2}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}v_{z_n}\bar{\gamma}_m\ rdrdz$$
$$-C_3\int_\Omega\frac{2}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}v_{z_n}\frac{\partial\bar{\gamma}_m}{\partial t}rdrdz-\int_\Omega F_2v_{z_n}rdrdz-C_2\int_0^1\frac{1}{\hat{\gamma}}F_1\bar{\gamma}_mdz-C_3\int_0^1\frac{1}{\hat{\gamma}}F_1\frac{\partial\bar{\gamma}_m}{\partial t}dz \tag{4.14}$$

STEP 2. Now we estimate the terms on the right hand-side of (4.14) by using the Cauchy's inequality, the properties of the space $\hat{\Gamma}$, and the assumption that $\hat{\gamma}$ is bounded away from zero, i.e., $\min_{z,t}\hat{\gamma}(z,t)\geq\delta>0$.

In particular, the first term on the righ hand-side of (4.14) is estimated as follows:

$$\left|\int_\Omega\frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial v_{z_n}}{\partial r}v_{z_n}\ r^2drdz\right|\leq\frac{C_1}{2}\int_\Omega\left|\frac{1}{\hat{\gamma}}\frac{\partial v_{z_n}}{\partial r}\right|^2\ rdrdz+\frac{1}{2C_1}\left\|\frac{\partial\hat{\gamma}}{\partial t}\right\|^2_C\int_\Omega\left|v_{z_n}\right|^2\ rdrdz.$$

The second and third terms on the righ hand-side of (4.14) satisfy

$$\frac{C_2}{2}\left|\int_0^1\frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}\left|\bar{\gamma}_m\right|^2\ dz\right|\leq\frac{C_2}{2\delta}\left\|\frac{\partial\hat{\gamma}}{\partial t}\right\|_C\int_0^1\left|\frac{1}{\sqrt{\hat{\gamma}}}\bar{\gamma}_m\right|^2\ dz,$$

and

$$C_2\left|\int_\Omega\frac{2}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}v_{z_n}\bar{\gamma}_m\ rdrdz\right|\leq C_2\left\|\frac{\partial\hat{\gamma}}{\partial z}\right\|^2_C\int_\Omega\left|v_{z_n}\right|^2\ rdrdz+\frac{C_2}{2\delta}\int_0^1\left|\frac{1}{\sqrt{\hat{\gamma}}}\bar{\gamma}_m\right|^2\ dz.$$

The 4-th term gives

$$C_3\left|\int_\Omega\frac{2}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}v_{z_n}\frac{\partial\bar{\gamma}_m}{\partial t}\ rdrdz\right|\leq\frac{2C_3}{\delta}\left\|\frac{\partial\hat{\gamma}}{\partial z}\right\|^2_C\int_\Omega\left|v_{z_n}\right|^2\ rdrdz+\frac{C_3}{4}\int_0^1\left|\frac{1}{\sqrt{\hat{\gamma}}}\frac{\partial\bar{\gamma}_m}{\partial t}\right|^2\ dz$$

In the remaining three terms we separate the source functions from the unknown functions, as usual, by performing the following estimates

$$\left|\int_\Omega F_2 v_{z_n}\ rdrdz\right| \le \frac{1}{4}\int_0^1 |F_2|^2\ dz + \frac{1}{2}\int_\Omega |v_{z_n}|^2\ rdrdz,$$

$$\left|C_2\int_0^1 \frac{1}{\hat\gamma}F_1\bar\gamma_m\ dz\right| \le \frac{C_2}{2}\int_0^1 |F_1|^2\ dz + \frac{C_2}{2\delta}\int_0^1 \left|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right|^2\ dz,$$

$$\left|C_3\int_0^1 \frac{1}{\hat\gamma}F_1\frac{\partial\bar\gamma_m}{\partial t}\ dz\right| \le \frac{C_3}{\delta}\int_0^1 |F_1|^2\ dz + \frac{C_3}{4}\int_0^1 \left|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial t}\right|^2\ dz.$$

These estimates combined give a basis for the Gronwall's inequality. Namely, we now have

$$\frac{1}{2}\frac{d}{dt}\left\{\|v_{z_n}\|^2_{L^2(\Omega,r)} + C_2\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^2(0,1)}\right\} + \frac{C_1}{2}\left\|\frac{1}{\hat\gamma}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)} + \frac{C_3}{2}\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial t}\right\|^2_{L^2(0,1)}$$
$$\le \frac{1}{2}\left(\frac{1}{C_1}\left\|\frac{\partial\hat\gamma}{\partial t}\right\|^2_C + \left(2+\frac{8}{C_3\delta}\right)\left\|\frac{\partial\hat\gamma}{\partial z}\right\|^2_C + 1\right)\int_\Omega |v_{z_n}|^2\ rdrdz \tag{4.15}$$
$$+\frac{1}{2}\left(\frac{2}{\delta}\left\|\frac{\partial\hat\gamma}{\partial t}\right\|_C + \frac{2}{C_2\delta} + \frac{1}{\delta}\right)C_2\int_0^1\left|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right|^2 dz + \frac{1}{4}\int_0^1 |F_2|^2\,dz + \left(\frac{C_2}{2}+\frac{C_3}{\delta}\right)\int_0^1 |F_1|^2\,dz$$

STEP 3. In this step we use Gronwall's inequality to estimate the L^2 norms of v_{z_n} and $\bar\gamma_m$. First, introduce the following notation:

$$\left\{\begin{array}{l}
Y(t) = \|v_{z_n}\|^2_{L^2(\Omega,r)} + C_2\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^2(0,1)}\\
A(t) = C_1\left\|\frac{1}{\hat\gamma}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)} + C_3\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial t}\right\|^2_{L^2(0,1)}\\
g(t) = \frac{1}{C_1}\left\|\frac{\partial\hat\gamma}{\partial t}\right\|^2_C + \left(2C_2+\frac{4C_3}{\delta}\right)\left\|\frac{\partial\hat\gamma}{\partial z}\right\|^2_C + 1 + \frac{2}{\delta}\left\|\frac{\partial\hat\gamma}{\partial t}\right\|_C + \frac{2}{C_2\delta} + \frac{1}{\delta}\\
D(t) = \frac{1}{4}\int_0^1 |F_2|^2 dz + (C_2 + \frac{2C_3}{\delta})\int_0^1 |F_1|^2 dz
\end{array}\right. \tag{4.16}$$

Then, inequality (4.15) takes the form

$$\frac{d}{dt}Y(t) + A(t) \le g(t)\ Y(t) + D(t). \tag{4.17}$$

The following version of Gronwall's inequality can now be employed:

Lemma 4.4 *(Gronwall, [12]) Let $Y(\cdot)$ be a non-negative, absolutely continuous function on $[0,T]$ and $g(\cdot), A(\cdot), D(\cdot)$ are non-negative, summable functions on $[0,T]$ such that for a.e. t in $[0,T]$ the following differential inequality holds*

$$Y'(t) + A(t) \leq g(t)Y(t) + D(t).$$

Then for all $t \in [0,T]$,

$$Y(t) + \int_0^t A(s)ds \leq \left[Y(0) + \int_0^t D(s)ds\right] \exp(\int_0^t g(s)ds).$$

Lemma 4.4 implies that for all $0 \leq t \leq T$

$$\sup_{0\leq t\leq T} Y(t) + \int_0^T A(t)dt \leq \left[\underbrace{Y(0)}_{init.\ data} + \underbrace{\int_0^T D(t)dt}_{bound.\ data}\right] \exp\left(\underbrace{\int_0^T g(t)dt}_{term\ involving\ \hat{\gamma}}\right).$$

From the form of $D(t)$, F_1 and F_2 we get

$$\int_0^T D(t)dt \leq \tilde{C}\left(\|\gamma_1 - \gamma_0\|^2_{H^1(0,T)} + \|\gamma_0'\|^2_{L^2(0,T)}\right),$$

where constant $\tilde{C}$ depends on C_2, C_3 and δ.

From this estimate and from the form of $g(t)$ we combine constant $\tilde{C}$ and the exponential term involving $\hat{\gamma}$ to define a constant C depending on the coefficients of (2.2)-(2.6) via $1/C_1$, C_2, C_3, $1/\delta$ where $\delta = \min_{z,t}\hat{\gamma}$, and $\|\hat{\gamma}\|_{H^1(0,T:C[0,1])}$, $\|\hat{\gamma}\|_{L^2(0,T:C^1[0,1])}$, such that

$$\sup_{0\leq t\leq T}\left\{\|v_{z_n}\|^2_{L^2(\Omega,r)} + \frac{C_2}{\hat{\gamma}_{\max}}\|\bar{\gamma}_m\|^2_{L^2(0,1)}\right\} + \frac{C_1}{\hat{\gamma}^2_{\max}}\left\|\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(0,T:L^2(\Omega,r))} + \frac{C_3}{\hat{\gamma}_{\max}}\left\|\frac{\partial \bar{\gamma}_m}{\partial t}\right\|^2_{L^2(0,T:L^2(0,1))}$$

$$\leq C\left(\|v_z^0\|^2_{L^2(\Omega,r)} + \|\bar{\gamma}^0\|^2_{L^2(0,1)} + \|\gamma_1 - \gamma_0\|^2_{H^1(0,T)} + \|\gamma_0'\|^2_{L^2(0,T)}\right). \tag{4.18}$$

From the boundedness of $\|\partial\bar{\gamma}_m/\partial t\|_{L^2(0,T:L^2(0,1))}$ and the conservation of mass equation we see that

$$\frac{\partial}{\partial z}\int_0^1 v_{z_n} r dr \in L^2(0,T:L^2(0,1)),$$

and its $L^2(0,T:L^2(0,1))$-norm is bounded by the right hand-side of the energy estimate (4.10).

STEP 4. We conclude the proof of Theorem 4.3 by showing that

$$\frac{\partial v_{z_n}}{\partial t} \in L^2(0,T : H^{-1}_{0,0}(\Omega, r)),$$

and that $\frac{\partial v_{z_n}}{\partial t}$ satisfies the estimate (4.10). Fix any $u \in H^1_{0,0}(\Omega, r)$ such that $\|u\|_{H^1_{0,0}(\Omega,r)} \leq 1$ and write $u = u_1 + u_2$, where $u_1 \in \operatorname{span}\{w_j\}_{j=1}^n$ and $(u_2, w_j)_{L^2} = 0$ for $j = 1, \cdots, n$. Then (4.2) and (4.4) imply

$$\int_\Omega \frac{\partial v_{z_n}}{\partial t} u \; rdrdz = \int_\Omega \frac{\partial v_{z_n}}{\partial t} u_1 \; rdrdz = -C_1 \int_\Omega \frac{1}{\hat{\gamma}^2} \frac{\partial v_{z_n}}{\partial r} \frac{\partial u_1}{\partial r} \; rdrdz$$
$$+ \underbrace{\int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} \frac{\partial v_{z_n}}{\partial r} u_1 \; r^2 drdz}_{(a)} + C_2 \int_0^1 \bar{\gamma}_m \frac{\partial}{\partial z} \int_0^1 u_1 \; rdr \; dz,$$
$$+ C_3 \int_0^1 \frac{\partial \bar{\gamma}_m}{\partial t} \frac{\partial}{\partial z} \int_0^1 u_1 \; rdr \; dz - \int_\Omega F_2 u_1 \; rdrdz.$$

Note that

$$(a) = -\int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} v_{z_n} \frac{\partial u_1}{\partial r} \; r^2 drdz - 2 \int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} v_{z_n} u_1 \; rdrdz$$

This implies

$$\left| \int_\Omega \frac{\partial v_{z_n}}{\partial t} u \; rdrdz \right| \leq \left[\frac{C_1}{\delta} \left\| \frac{1}{\hat{\gamma}} \frac{\partial v_{z_n}}{\partial r} \right\|_{L^2(\Omega,r)} + \frac{3}{\delta} \left\| \frac{\partial \hat{\gamma}}{\partial t} \right\|_{C[0,1]} \|v_{z_n}\|_{L^2(\Omega,r)} \right.$$
$$\left. + C_2 \|\bar{\gamma}_m\|_{L^2} + C_3 \left\| \frac{\partial \bar{\gamma}_m}{\partial t} \right\|_{L^2} + |F_2(t)| \right] \|u_1\|_{H^1_{0,0}(\Omega,r)} .$$

Thus, since $\|u_1\|_{H^1_{0,0}(\Omega,r)} \leq 1$ and $\left(\sum_{i=1}^5 x_i \right)^2 \leq 5 \sum_{i=1}^5 x_i^2$ we obtain, using the energy estimate (4.18):

$$\int_0^T \left\| \frac{\partial v_{z_n}}{\partial t} \right\|^2_{H^{-1}_{0,0}(\Omega,r)} \leq C \left(\|v_z^0\|^2_{L^2(\Omega,r)} + \|\bar{\gamma}^0\|^2_{L^2(0,1)} + \|\gamma_1 - \gamma_0\|^2_{H^1(0,T)} + \|\gamma_0'\|^2_{L^2(0,T)} \right).$$

This concludes the proof of Theorem 4.3. ■

It is interesting to notice that our energy estimate blows up when the coefficient of fluid viscosity C_1 approaches zero, and when the minimum δ of $\hat{\gamma}$ approaches zero, which corresponds to having the "linearized" vessel radius approaching zero. Both of these are reasonable. Additionally, notice that the coefficient of the vessel wall viscosity, C_3, governs the estimate for the time-derivative of the structure displacement $\partial \bar{\gamma}_m / \partial t$, which is also to be expected. Our estimate shows how structure viscoelasticity regularizes the time evolution of the structure motion.

Finally, notice that the right hand-side of the estimate incorporates the initial data for both the velocity and the structure and the boundary data for only the structure. This is consistent with the problem and is an interesting feature of the reduced, effective model studied in this manuscript.

4.3 Existence of a Weak Solution

Now we use the energy estimate to pass to the limits in the Galerkin approximations to obtain

Theorem 4.5 *There exists a weak solution of* (2.2)-(2.6).

Proof: We use the uniform bounds obtained by the energy estimate to conclude that there exist convergent subsequences that converge weakly to the functions which satisfy (2.2)-(2.4) in the weak sense. This is a standard approach except for the fact that we need to deal with the weighted L^2-norms in Ω, with the weight r that is present due to the axial symmetry of the problem. We deal with this technical obstacle by using the following Lemma, [1], with $p = 2$ and $\nu = 1$.

Lemma 4.6 *[1] If $\nu > 0$, $p \geq 1$, and $u \in C^1(0,R)$ then*

$$\int_0^R |u(r)|^p \, r^{\nu-1} dr \leq \frac{\nu+1}{\nu T} \int_0^R |u(r)|^p \, r^\nu dr + \frac{p}{\nu} \int_0^R |u(r)|^{p-1} \, |u'(r)| \, r^\nu dr.$$

By the energy estimate (4.10) we see that the sequence $\{\bar{\gamma}_m\}_{m=1}^\infty$ is bounded in $H^1(0,T : L^2(0,1))$. Similarly, $\{v_{z_n}\}_{n=1}^\infty$ is bounded in $L^2(0,T : H^1_{0,0}(\Omega, r))$ and that $\partial v_{z_n}/\partial t$ is bounded in $L^2(0,T : H^{-1}_{0,0}(\Omega, r)$.

Therefore, there exist convergent subsequences $\left\{\bar{\gamma}_{m_j}\right\}_{m_j=1}^\infty$ and $\left\{v_{z_{n_j}}\right\}_{n_j=1}^\infty$ such that

$$\begin{cases} \gamma_{m_j} \rightharpoonup \gamma & \text{weakly in } H^1(0,T : L^2(0,1)), \\ v_{z_{n_j}} \rightharpoonup v_z & \text{weakly in } L^2(0,T : L^2(\Omega, r)), \\ \dfrac{\partial v_{z_{n_j}}}{\partial r} \rightharpoonup \dfrac{\partial v_z}{\partial r} & \text{weakly in } L^2(0,T : L^2(\Omega, r)), \\ \dfrac{\partial v_{z_{n_j}}}{\partial t} \rightharpoonup \dfrac{\partial v_z}{\partial t} & \text{weakly in } L^2(0,T : H^{-1}_{0,0}(\Omega, r)). \end{cases} \tag{4.19}$$

We now show that the limiting function $(\bar{\gamma}, v_z)$ is a weak solution to (2.2)-(2.6). Fix two integers M and N and consider the functions $\Phi \in C^1([0,T] : H^1_0(0,1))$ and $\mathbf{w} \in C^1([0,T] : H^1_{0,0}(\Omega, r))$ of the form

$$\Phi(t) = \sum_{k=1}^M d_k(t)\phi_k, \quad \mathbf{w}(t) = \sum_{p=1}^N l_p(t) w_p,$$

where $\{d_k\}_{k=1}^M$ and $\{l_p\}_{p=1}^N$ are smooth functions. Let $n \geq N$ and $m \geq M$. Multiply (4.3) and (4.4), written in terms of the subsequences of $\bar{\gamma}_m$ and v_{z_n}, by $d_k(t)$, $l_p(t)$, sum over k and p for $k = 1, \cdots, N$ and $p = 1, \cdots, M$ and then integrate over $(0,T)$ with respect to t to obtain

$$\begin{aligned} &\int_0^T \int_0^1 \frac{1}{\hat{\gamma}} \frac{\partial \bar{\gamma}_{m_j}}{\partial t} \Phi \; dzdt + \int_0^T \int_\Omega \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} v_{z_{n_i}} \Phi \; rdrdzdt \\ &- \int_0^T \int_0^1 \int_0^1 v_{z_{n_i}} \; rdr \; \frac{\partial \Phi}{\partial z} \; dzdt + \int_0^T \int_0^1 \frac{1}{\hat{\gamma}} F_1 \Phi \; dzdt = 0, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial v_{z_{n_i}}}{\partial t} \mathbf{w} \; rdrdzdt + C_1 \int_0^T \int_\Omega \frac{1}{\hat{\gamma}^2} \frac{\partial v_{z_{n_i}}}{\partial r} \frac{\partial \mathbf{w}}{\partial r} \; rdrdzdt \\ &- \int_0^T \int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} \frac{\partial v_{z_{n_i}}}{\partial r} \mathbf{w} \; r^2 drdzdt = C_2 \int_0^T \int_0^1 \bar{\gamma}_{m_j} \frac{\partial}{\partial z} \int_0^1 \mathbf{w} \; rdr \; dzdt \\ &+ C_3 \int_0^T \int_0^1 \frac{\partial \bar{\gamma}_{m_j}}{\partial t} \frac{\partial}{\partial z} \int_0^1 \mathbf{w} \; rdr \; dzdt - \int_0^T \int_\Omega F_2 \mathbf{w} \; rdrdzdt. \end{aligned} \tag{4.21}$$

To pass to the weak limit as $i, j \to \infty$ we use the fact that $\hat{\gamma} \in H^1(0, T : C^1[0,1])$. Equation (4.19) implies that in the limit the following holds

$$\begin{aligned} &\int_0^T \int_0^1 \frac{1}{\hat{\gamma}} \frac{\partial \bar{\gamma}}{\partial t} \Phi \; dzdt + \int_0^T \int_\Omega \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} v_z \Phi \; rdrdzdt \\ &- \int_0^T \int_0^1 \int_0^1 v_z \; rdr \; \frac{\partial \Phi}{\partial z} \; dzdt + \int_0^T \int_0^1 \frac{1}{\hat{\gamma}} F_1 \Phi \; dzdt = 0, \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial v_z}{\partial t} \mathbf{w} \; rdrdzdt + C_1 \int_0^T \int_\Omega \frac{1}{\hat{\gamma}^2} \frac{\partial v_z}{\partial r} \frac{\partial \mathbf{w}}{\partial r} \; rdrdzdt \\ &- \int_0^T \int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} \frac{\partial v_z}{\partial r} \mathbf{w} \; r^2 drdzdt = C_2 \int_0^T \int_0^1 \bar{\gamma} \frac{\partial}{\partial z} \int_0^1 \mathbf{w} \; rdr \; dz \\ &+ C_3 \int_0^T \int_0^1 \frac{\partial \bar{\gamma}}{\partial t} \frac{\partial}{\partial z} \int_0^1 \mathbf{w} \; rdr \; dzdt - \int_0^T \int_\Omega F_2 \mathbf{w} \; rdrdzdt. \end{aligned} \tag{4.23}$$

These equations hold for all the functions $\Phi \in L^2(0, T : H_0^1(0,1))$ and $\mathbf{w} \in L^2(0, T : H_{0,0}^1(\Omega, r))$ since $C^1([0,T] : H_0^1(0,1))$ and $C^1([0,T] : H_{0,0}^1(\Omega, r))$ are dense in $L^2(0, T : H_0^1(0,1))$ and $L^2(0, T : H_{0,0}^1(\Omega, r))$, respectively. This implies that for all $\phi \in H_0^1(0,1)$ and $w \in H_{0,0}^1(\Omega, r)$ and a.e $0 \leq t \leq T$ the weak form of (2.2)-(2.4) is satisfied

$$\int_0^1 \frac{1}{\hat{\gamma}} \frac{\partial \bar{\gamma}}{\partial t} \phi \; dz + \int_\Omega \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} v_z \phi \; rdrdz - \int_0^1 \int_0^1 v_z \; rdr \; \frac{\partial \phi}{\partial z} \; dz + \int_0^1 \frac{1}{\hat{\gamma}} F_1 \phi \; dz = 0, \tag{4.24}$$

and

$$\int_\Omega \frac{\partial v_z}{\partial t} w \ rdrdz + C_1 \int_\Omega \frac{1}{\hat{\gamma}^2} \frac{\partial v_z}{\partial r} \frac{\partial w}{\partial r} \ rdrdz - \int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial t} \frac{\partial v_z}{\partial r} w \ r^2 drdz$$
$$= C_2 \int_0^1 \bar{\gamma} \frac{\partial}{\partial z} \int_0^1 w \ rdr \ dz \ + C_3 \int_0^1 \frac{\partial \bar{\gamma}}{\partial t} \frac{\partial}{\partial z} \int_0^1 w \ rdr \ dz - \int_\Omega F_2 w \ rdrdz. \tag{4.25}$$

Furthermore, equation (4.24) implies that

$$\frac{\partial}{\partial z} \int_0^1 v_z \ rdr = -\frac{1}{\hat{\gamma}} \frac{\partial \bar{\gamma}}{\partial t} - \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} \int_0^1 v_z \ rdr - \frac{1}{\hat{\gamma}} F_1 \text{ in the weak sense,}$$

and so

$$\frac{\partial}{\partial z} \int_0^1 v_z \ rdr \in L^2(0,1) \ a.e. \ t \in [0,T],$$

and consequently $\frac{\partial}{\partial z} \int_0^1 v_z \ rdr \in L^2(0,T : L^2(0,1))$.

To check that the limiting functions satisfy the initial data we proceed as follows. Let $\Phi \in C^1([0,T] : H_0^1(0,L))$ with $\Phi(T) = 0$. Integrate (4.22) by parts once with respect to t to obtain

$$-\int_0^T \int_0^1 \frac{1}{\hat{\gamma}} \bar{\gamma} \frac{\partial \Phi}{\partial t} \ dzdt + \int_0^T \int_0^1 \frac{1}{\hat{\gamma}^2} \frac{\partial \hat{\gamma}}{\partial t} \bar{\gamma} \Phi \ dzdt + \int_0^T \int_\Omega \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} v_z \Phi \ rdrdzdt,$$
$$-\int_0^T \int_0^1 \int_0^1 v_z \ rdr \ \frac{\partial \Phi}{\partial z} \ dzdt + \int_0^T \int_0^1 \frac{1}{\hat{\gamma}} F_1 \Phi \ dzdt - \int_0^1 \left[\frac{1}{\hat{\gamma}} \bar{\gamma} \Phi \right]_{t=0} \ dz = 0.$$

Similarly from (4.20) we deduce

$$-\int_0^T \int_0^1 \frac{1}{\hat{\gamma}} \bar{\gamma}_m \frac{\partial \Phi}{\partial t} \ dzdt + \int_0^T \int_0^1 \frac{1}{\hat{\gamma}^2} \frac{\partial \hat{\gamma}}{\partial t} \bar{\gamma}_m \Phi \ dzdt + \int_0^T \int_\Omega \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} v_{z_n} \Phi \ rdrdzdt,$$
$$-\int_0^T \int_0^1 \int_0^1 v_{z_n} \ rdr \ \frac{\partial \Phi}{\partial z} \ dzdt + \int_0^T \int_0^1 \frac{1}{\hat{\gamma}} F_1 \Phi \ dzdt - \int_0^1 \left[\frac{1}{\hat{\gamma}} \bar{\gamma}_m \Phi \right]_{t=0} \ dz = 0.$$

Set $m = m_j$ in the above equation, and let $m_j \to \infty$. Since $\Phi(0)$ is arbitrary, and because of the convergence (4.19) and the initial data (2.4) we conclude that $\bar{\gamma}_m$ converges weakly to a function $\bar{\gamma}$ which satisfies

$$\bar{\gamma}(z,0) = \bar{\gamma}^0(z).$$

A similar approach verifies the initial data for the limiting function v_z.

Therefore $(\bar{\gamma}, v_z)$ is a weak solution of (2.2)-(2.6).

$\blacksquare$

Corollary 4.7 *From the energy estimate* (4.10) *we see that, in fact,*

$$\bar{\gamma} \in L^\infty(0,T : L^2(0,1)) \cap H^1(0,T : L^2(0,1)), \tag{4.26}$$

$$v_z \in L^2(0,T : H^1_{0,0}(\Omega, r)) \cap L^\infty(0,T : L^2(\Omega, r)) \text{ with } \frac{\partial v_z}{\partial t} \in L^2(0,T : H^{-1}_{0,0}(\Omega, r)). \tag{4.27}$$

4.4 Uniqueness of a Weak Solution

Energy estimate (4.10) implies the following result

Theorem 4.8 *A weak solution of* (2.2)-(2.6) *is unique.*

Proof: By setting $\psi = \hat{\gamma}$ and $w = v_z$ in the weak form (3.4), (3.5), and by using the Gronwall's inequality as in the energy estimate with zero initial and boundary data, see [12], uniqueness of a weak solution is established. ■

Theorems 4.5 and 4.8 imply the existence of a unique weak solution to problem (2.2)-(2.6), namely Theorem 4.1.

5 Improved Regularity

We now show that the sequence $\left\{\frac{\partial v_{zn}}{\partial t}\right\}_{n=1}^\infty$ is bounded in $L^2(0,T : L^2(\Omega, r))$. More precisely, we prove the following

Theorem 5.1 **(Improved Regularity: Part I)** *Suppose that the coefficient function $\hat{\gamma} \in \hat{\Gamma}$, and that the initial data $\bar{\gamma}^0 \in L^2(0,1)$, $v_z^0 \in H^1_{0,0}(\Omega, r)$ and the boundary data $\gamma_0, \gamma_1 \in H^2(0,T)$. Suppose also that $(\bar{\gamma}, v_z) \in \Gamma \times V$ is a weak solution of* (2.2)-(2.6). *Then, in fact,*

$$\frac{\partial v_z}{\partial t} \in L^2(0,T : L^2(\Omega, r)), \frac{\partial \bar{\gamma}}{\partial t} \in L^\infty(0,T : L^2(0,1)), \frac{\partial v_z}{\partial r} \in L^\infty(0,T : L^2(\Omega, r)),$$

and the following estimate holds

$$\begin{aligned} &\text{ess sup}_{0 \le t \le T} \left\{ \left\| \frac{1}{\sqrt{\bar{\gamma}}} \frac{\partial \bar{\gamma}}{\partial t} \right\|^2_{L^2(0,1)} + \left\| \frac{1}{\hat{\gamma}} \frac{\partial v_z}{\partial r} \right\|^2_{L^2(\Omega, r)} \right\} + \left\| \frac{\partial v_z}{\partial t} \right\|^2_{L^2(0,T:L^2(\Omega,r))} \\ \le & C \left(\|\bar{\gamma}^0\|^2_{L^2(0,1)} + \|v_z^0\|^2_{H^1_{0,0}(\Omega,r)} + \|\gamma_1 - \gamma_0\|^2_{H^2(0,T)} + \|\gamma_0\|^2_{H^2(0,T)} \right). \end{aligned}$$

Proof: Similarly as before, we need to get rid off the terms that we cannot control at this point, namely, the terms involving the z-derivatives of $\bar{\gamma}_m$. In order to do this we manipulate the conservation of mass and momentum equations, add them up to cancel the unwanted terms, and obtain an equation which we can estimate using the Cauchy's and Young's inequalities. The final estimate is then obtained by an application of the Gronwall's inequality.

Thus, we begin by multiplying (4.4) by $\dot{l}_k^n(t)$, and summing from $k = 1, \cdots, n$ to find

$$
\begin{aligned}
&\int_\Omega \left|\frac{\partial v_{z_n}}{\partial t}\right|^2 rdrdz + \frac{C_1}{2}\frac{d}{dt}\int_\Omega \frac{1}{\hat{\gamma}^2}\left|\frac{\partial v_{z_n}}{\partial r}\right|^2 rdrdz + C_1\int_\Omega \frac{1}{\hat{\gamma}^3}\frac{\partial\hat{\gamma}}{\partial t}\left|\frac{\partial v_{z_n}}{\partial r}\right|^2 rdrdz \\
&-\int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial v_{z_n}}{\partial r}\frac{\partial v_{z_n}}{\partial t}r^2 drdz = -\underbrace{C_2\int_\Omega \frac{\partial\bar{\gamma}_m}{\partial z}\frac{\partial v_{z_n}}{\partial t}rdrdz}_{(a)} - \underbrace{C_3\int_\Omega \frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\frac{\partial v_{z_n}}{\partial t}rdrdz}_{(b)} \\
&-\int_\Omega F_2\frac{\partial v_{z_n}}{\partial t}\, rdrdz. \qquad (5.1)
\end{aligned}
$$

Here we used integration by parts with respect to z in the first and second term of the right-hand side of the equality.

Next, differentiate (4.3) with respect to t and multiply by $C_3\dot{d}_h^m(t)$, and sum $h = 1, \cdots, m$ to find

$$
\begin{aligned}
&\frac{C_3}{2}\frac{d}{dt}\int_0^1 \frac{1}{\hat{\gamma}}\left|\frac{\partial\bar{\gamma}_m}{\partial t}\right|^2 dz - \frac{C_3}{2}\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}\left|\frac{\partial\bar{\gamma}_m}{\partial t}\right|^2 dz - C_3\int_\Omega \frac{2}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial\hat{\gamma}}{\partial z}v_{z_n}\frac{\partial\bar{\gamma}_m}{\partial t}\, rdrdz \\
&+C_3\int_\Omega \frac{2}{\hat{\gamma}}\frac{\partial^2\hat{\gamma}}{\partial z\partial t}v_{z_n}\frac{\partial\bar{\gamma}_m}{\partial t}\, rdrdz + C_3\int_\Omega \frac{2}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}\frac{\partial v_{z_n}}{\partial t}\frac{\partial\bar{\gamma}_m}{\partial t}\, rdrdz = \underbrace{C_3\int_\Omega \frac{\partial v_{z_n}}{\partial t}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}rdrdz}_{(b)} \\
&+C_3\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}F_1\frac{\partial\bar{\gamma}_m}{\partial t}dz - C_3\int_0^1 \frac{1}{\hat{\gamma}}\frac{\partial F_1}{\partial t}\frac{\partial\bar{\gamma}_m}{\partial t}dz. \qquad (5.2)
\end{aligned}
$$

Finally, differentiate (4.3) with respect to t and multiply by $C_2 d_h^m(t)$, and sum $h = 1, \cdots, m$ to find

$$
\begin{aligned}
&C_2\int_0^1 \frac{1}{\hat{\gamma}}\frac{\partial^2\bar{\gamma}_m}{\partial t^2}\bar{\gamma}_m\, dz - C_2\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial\bar{\gamma}_m}{\partial t}\bar{\gamma}_m\, dz - C_2\int_\Omega \frac{2}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial\hat{\gamma}}{\partial z}v_{z_n}\bar{\gamma}_m\, rdrdz \\
&+C_2\int_\Omega \frac{2}{\hat{\gamma}}\frac{\partial^2\hat{\gamma}}{\partial z\partial t}v_{z_n}\bar{\gamma}_m\, rdrdz + C_2\int_\Omega \frac{2}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}\frac{\partial v_{z_n}}{\partial t}\bar{\gamma}_m\, rdrdz = \underbrace{C_2\int_\Omega \frac{\partial v_{z_n}}{\partial t}\frac{\partial\bar{\gamma}_m}{\partial z}rdrdz}_{(a)} \\
&+C_2\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial t}F_1\bar{\gamma}_m dz - C_2\int_0^1 \frac{1}{\hat{\gamma}}\frac{\partial F_1}{\partial t}\bar{\gamma}_m dz. \qquad (5.3)
\end{aligned}
$$

By adding (5.1) through (5.3) we obtain:

$$\frac{d}{dt}\left[\frac{C_3}{2}\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial t}\right\|^2_{L^2(0,1)}+\frac{C_1}{2}\left\|\frac{1}{\hat\gamma}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)}\right]+\left\|\frac{\partial v_{z_n}}{\partial t}\right\|^2_{L^2(\Omega,r)}$$
$$= -C_1\int_\Omega \frac{1}{\hat\gamma^3}\frac{\partial\hat\gamma}{\partial t}\left|\frac{\partial v_{z_n}}{\partial r}\right|^2 rdrdz+\int_\Omega\frac{1}{\hat\gamma}\frac{\partial\hat\gamma}{\partial t}\frac{\partial v_{z_n}}{\partial r}\frac{\partial v_{z_n}}{\partial t}r^2drdz+\frac{C_3}{2}\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial t}\left|\frac{\partial\bar\gamma_m}{\partial t}\right|^2dz$$
$$-2C_3\int_\Omega\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial t}\frac{\partial\hat\gamma}{\partial z}v_{z_n}\frac{\partial\bar\gamma_m}{\partial t}\ rdrdz-2C_3\int_\Omega\frac{1}{\hat\gamma}\frac{\partial^2\hat\gamma}{\partial z\partial t}v_{z_n}\frac{\partial\bar\gamma_m}{\partial t}\ rdrdz$$
$$+2C_3\int_\Omega\frac{1}{\hat\gamma}\frac{\partial\hat\gamma}{\partial z}\frac{\partial v_{z_n}}{\partial t}\frac{\partial\bar\gamma_m}{\partial t}\ rdrdz-C_2\int_0^1\frac{1}{\hat\gamma}\frac{\partial^2\bar\gamma_m}{\partial t^2}\bar\gamma_m\ dz+C_2\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial t}\frac{\partial\bar\gamma_m}{\partial t}\bar\gamma_m\ dz$$
$$+2C_2\int_\Omega\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial t}\frac{\partial\hat\gamma}{\partial z}v_{z_n}\bar\gamma_m\ rdrdz-2C_2\int_\Omega\frac{1}{\hat\gamma}\frac{\partial^2\hat\gamma}{\partial z\partial t}v_{z_n}\bar\gamma_m\ rdrdz$$
$$-2C_2\int_\Omega\frac{1}{\hat\gamma}\frac{\partial\hat\gamma}{\partial z}\frac{\partial v_{z_n}}{\partial t}\bar\gamma_m\ rdrdz-\int_\Omega F_2\frac{\partial v_{z_n}}{\partial t}\ rdrdz+C_3\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial t}F_1\frac{\partial\bar\gamma_m}{\partial t}dz$$
$$-C_3\int_0^1\frac{1}{\hat\gamma}\frac{\partial F_1}{\partial t}\frac{\partial\bar\gamma_m}{\partial t}dz+C_2\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial t}F_1\bar\gamma_m dz-C_2\int_0^1\frac{1}{\hat\gamma}\frac{\partial F_1}{\partial t}\bar\gamma_m\ dz. \tag{5.4}$$

Notice that terms (a) and (b) cancelled out.

Before we estimate the right hand-side of this equation, we will integrate the entire equation with respect to t in order to be able to deal with the term on the right hand-side of this equation, which contains the second derivative with respect to t of $\bar\gamma_m$. This term will then be integrated by parts with respect to t to obtain the terms which can be estimated using the information that we already have at this point. More precisely, the term

$$C_2\int_0^t\int_0^1\frac{1}{\hat\gamma}\frac{\partial^2\bar\gamma_m}{\partial t^2}\bar\gamma_m\ dzds$$

integrated by parts with respect to t gives

$$-C_2\int_0^t\int_0^1\frac{1}{\hat\gamma}\frac{\partial^2\bar\gamma_m}{\partial s^2}\bar\gamma_m\ dzds=-C_2\int_0^t\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial s}\frac{\partial\bar\gamma_m}{\partial s}\bar\gamma_m\ dzds+C_2\int_0^t\int_0^1\frac{1}{\hat\gamma}\left|\frac{\partial\bar\gamma_m}{\partial s}\right|^2\ dz$$
$$-C_2\int_0^1\frac{1}{\hat\gamma}\frac{\partial\bar\gamma_m(z,t)}{\partial t}\bar\gamma_m(z,t)\ dz+C_2\int_0^1\frac{1}{\hat\gamma(z,0)}\frac{\partial\bar\gamma_m(z,0)}{\partial t}\bar\gamma_m(z,0)\ dz$$
$$=-C_2\int_0^t\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial s}\frac{\partial\bar\gamma_m}{\partial s}\bar\gamma_m\ dzds+C_2\int_0^t\int_0^1\frac{1}{\hat\gamma}\left|\frac{\partial\bar\gamma_m}{\partial s}\right|^2\ dz$$
$$-C_2\int_0^1\frac{1}{\hat\gamma}\frac{\partial\bar\gamma_m(z,t)}{\partial t}\bar\gamma_m(z,t)\ dz+C_2\int_0^1\frac{1}{\hat\gamma(z,0)}F_1(0)\bar\gamma_m(z,0)\ dz.$$

We use this equation to express the time integral over $(0,t)$ of (5.4) as follows:

$$
\begin{aligned}
&\frac{1}{2}\left\{C_3\left\|\frac{1}{\sqrt{\hat{\gamma}(t)}}\frac{\partial\bar{\gamma}_m(t)}{\partial t}\right\|^2_{L^2(0,1)} dz + C_1\left\|\frac{1}{\hat{\gamma}(t)}\frac{\partial v_{z_n}(t)}{\partial r}\right\|^2_{L^2(\Omega,r)}\right\} + \int_0^t \left\|\frac{\partial v_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)} ds \\
&= \frac{1}{2}\left\{C_3\left\|\frac{1}{\sqrt{\hat{\gamma}(0)}}\frac{\partial\bar{\gamma}_m(0)}{\partial t}\right\|^2_{L^2(0,1)} dz + C_1\left\|\frac{1}{\hat{\gamma}(0)}\frac{\partial v_{z_n}(0)}{\partial r}\right\|^2_{L^2(\Omega,r)}\right\} \\
&+ C_1\int_0^t\int_\Omega \frac{1}{\hat{\gamma}^3}\frac{\partial\hat{\gamma}}{\partial s}\left|\frac{\partial v_{z_n}}{\partial r}\right|^2 \, rdrdzds \\
&+ \int_0^t\int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial s}\frac{\partial v_{z_n}}{\partial r}\frac{\partial v_{z_n}}{\partial s} r^2 drdzds + \frac{C_3}{2}\int_0^t\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial s}\left|\frac{\partial\bar{\gamma}_m}{\partial s}\right|^2 dzds \\
&+ 2C_3\int_0^t\int_\Omega \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial s}\frac{\partial\hat{\gamma}}{\partial z} v_{z_n}\frac{\partial\bar{\gamma}_m}{\partial s} \, rdrdzds - 2C_3\int_0^t\int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial^2\hat{\gamma}}{\partial z\partial s} v_{z_n}\frac{\partial\bar{\gamma}_m}{\partial s} \, rdrdzds \\
&+ 2C_3\int_0^t\int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}\frac{\partial v_{z_n}}{\partial s}\frac{\partial\bar{\gamma}_m}{\partial s} \, rdrdzds + C_2\int_0^t\int_0^1 \frac{1}{\hat{\gamma}}\left|\frac{\partial\bar{\gamma}_m}{\partial s}\right|^2 \, dz \\
&- C_2\int_0^1 \frac{1}{\hat{\gamma}}\frac{\partial\bar{\gamma}_m(z,t)}{\partial t}\bar{\gamma}_m(z,t)\, dz + C_2\int_0^1 \frac{1}{\hat{\gamma}(z,0)} F_1(0)\bar{\gamma}_m(z,0)\, dz \\
&+ 2C_2\int_0^t\int_\Omega \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial s}\frac{\partial\hat{\gamma}}{\partial z} v_{z_n}\bar{\gamma}_m \, rdrdzds - 2C_2\int_0^t\int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial^2\hat{\gamma}}{\partial z\partial s} v_{z_n}\bar{\gamma}_m \, rdrdzds \\
&+ 2C_2\int_0^t\int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial z}\frac{\partial v_{z_n}}{\partial s}\bar{\gamma}_m \, rdrdzds - \int_0^t\int_\Omega F_2\frac{\partial v_{z_n}}{\partial s} \, rdrdzds \\
&+ C_3\int_0^t\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial s} F_1\frac{\partial\bar{\gamma}_m}{\partial s} dzds - C_3\int_0^t\int_0^1 \frac{1}{\hat{\gamma}}\frac{\partial F_1}{\partial s}\frac{\partial\bar{\gamma}_m}{\partial s} dzds \\
&+ C_2\int_0^t\int_0^1 \frac{1}{\hat{\gamma}^2}\frac{\partial\hat{\gamma}}{\partial s} F_1\bar{\gamma}_m dzds - C_2\int_0^t\int_0^1 \frac{1}{\hat{\gamma}}\frac{\partial F_1}{\partial s}\bar{\gamma}_m \, dzds.
\end{aligned}
$$

Now, the right hand-side is given in terms of the functions which can be estimated. The estimates are presented next.

We begin by estimating the first terms on the right hand-side, containing information at $t=0$. To estimate the term involving the time derivative of $\bar{\gamma}_m$, we use the weak form of the conservation of mass equation (4.3), multiply it by $C_3\dot{d}_h^m$, sum over $h = 1, \cdots, m$ and evaluate the resulting expression at $t = 0$. Then we use the fact that v_{z_n} converges weakly in L^2 to v_z, integrate by parts the term involving the second derivative of γ_m with respect to z and t, and use the Cauchy inequality with ϵ to obtain the following estimate

$$\begin{aligned}
&\int_0^1 \frac{1}{\hat{\gamma}(z,0)} \left|\frac{\partial \bar{\gamma}_m(z,0)}{\partial t}\right|^2 dz \le \int_0^1 \left| \frac{2}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial z} \frac{\partial \bar{\gamma}_m}{\partial t} \left(\int_0^1 v_z r dr \right) \right|_{t=0} dz \\
&+ \int_0^1 \left| \frac{\partial \bar{\gamma}_m}{\partial t} \frac{\partial}{\partial z} \left(\int_0^1 v_z \, r dr \right) \right|_{t=0} dz + C_3 \int_0^1 \left| \frac{1}{\hat{\gamma}} F_1 \frac{\partial \bar{\gamma}_m}{\partial t} \right|_{t=0} dz \\
&\le \frac{1}{4\epsilon} \frac{2}{\delta} \left\| \frac{\partial \hat{\gamma}}{\partial z}(0) \right\|_C^2 \left\| \int_0^1 v_z^0 \, r dr \right\|_{L^2(0,1)}^2 + \frac{1}{4\epsilon} \max_z \hat{\gamma}^2(0) \left\| \frac{\partial}{\partial z} \int_0^1 v_z^0 \, r dr \right\|_{L^2(0,1)}^2 \\
&+ \frac{1}{4\epsilon} \frac{1}{\delta} \left\| F_1(\cdot, 0) \right\|_{L^2(0,1)}^2 + 3\epsilon \int_0^1 \frac{1}{\hat{\gamma}(z,0)} \left| \frac{\partial \bar{\gamma}_m(z,0)}{\partial t} \right|^2 dz.
\end{aligned}$$

This implies that there exists a constant $\tilde{K} > 0$ depending on C_1, $\frac{1}{\delta}$ and $\|\hat{\gamma}(0)\|_{C^1[0,1]}$ such that

$$\int_0^1 \frac{1}{\hat{\gamma}(z,0)} \left| \frac{\partial \bar{\gamma}_m(z,0)}{\partial t} \right|^2 dz \le \tilde{K} \left\{ \|v_z^0\|_{H_{0,0}^1(\Omega,r)}^2 + (\gamma_1'(0) - \gamma_0'(0))^2 + (\gamma_0'(0))^2 \right\}.$$

This estimate, combined with

$$\left\| \frac{\partial v_{z_n}}{\partial r}(\cdot,\cdot,0) \right\|_{L^2(\Omega,r)} \le \left\| \frac{\partial v_z^0}{\partial r} \right\|_{L^2(\Omega,r)}$$

implies

$$\begin{aligned}
&C_3 \left\| \frac{1}{\sqrt{\hat{\gamma}(0)}} \frac{\partial \bar{\gamma}_m(0)}{\partial t} \right\|_{L^2(0,1)}^2 dz + C_1 \left\| \frac{1}{\hat{\gamma}(0)} \frac{\partial v_{z_n}(0)}{\partial r} \right\|_{L^2(\Omega,r)}^2 \\
&\le K \left\{ \|v_z^0\|_{H_{0,0}^1(\Omega,r)}^2 + (\gamma_1'(0) - \gamma_0'(0))^2 + (\gamma_0'(0))^2 \right\},
\end{aligned}$$

where $K > 0$ depends on $C_1, C_3, \frac{1}{\delta}$ and $\|\hat{\gamma}(0)\|_{C^1[0,1]}$.

The second, the third and the fourth terms are estimated as follows:

$$\left| C_1 \int_0^t \int_\Omega \frac{1}{\hat{\gamma}^3} \frac{\partial \hat{\gamma}}{\partial s} \left| \frac{\partial v_{z_n}}{\partial r} \right|^2 \, r dr dz ds \right| \le \frac{C_1}{\delta} \int_0^t \left\| \frac{\partial \hat{\gamma}}{\partial s} \right\|_C \left\| \frac{1}{\hat{\gamma}} \frac{\partial v_{z_n}}{\partial r} \right\|_{L^2(\Omega,r)}^2 ds,$$

$$\left| \int_0^t \int_\Omega \frac{1}{\hat{\gamma}} \frac{\partial \hat{\gamma}}{\partial s} \frac{\partial v_{z_n}}{\partial r} \frac{\partial v_{z_n}}{\partial s} r^2 dr dz ds \right| \le \frac{1}{4\epsilon_1} \int_0^t \left\| \frac{\partial \hat{\gamma}}{\partial s} \right\|_C^2 \left\| \frac{1}{\hat{\gamma}} \frac{\partial v_{z_n}}{\partial r} \right\|_{L^2(\Omega,r)}^2 ds + \epsilon_1 \int_0^t \left\| \frac{\partial v_{z_n}}{\partial s} \right\|_{L^2(\Omega,r)}^2 ds.$$

$$\left| \frac{C_3}{2} \int_0^t \int_0^1 \frac{1}{\hat{\gamma}^2} \frac{\partial \hat{\gamma}}{\partial s} \left| \frac{\partial \bar{\gamma}_m}{\partial s} \right|^2 dz ds \right| \le \frac{C_3}{2\delta} \int_0^t \left\| \frac{\partial \hat{\gamma}}{\partial s} \right\|_C \left\| \frac{1}{\sqrt{\hat{\gamma}}} \frac{\partial \bar{\gamma}_m}{\partial s} \right\|_{L^2(0,1)}^2 ds.$$

The estimates of the fifth and the sixth term make use of the L^∞ norm of v_{z_n}:

$$\left|2C_3\int_0^t\int_\Omega \frac{1}{\hat\gamma}\frac{\partial\hat\gamma}{\partial z}\frac{\partial\hat\gamma}{\partial s}v_{z_n}\frac{\partial\bar\gamma_m}{\partial s}\ rdrdzds\right|$$
$$\leq \frac{C_3}{\delta}\left\|v_{z_n}\right\|^2_{L^\infty(0,t:L^2(\Omega,r))}\left\|\frac{\partial\hat\gamma}{\partial t}\right\|^2_{L^2(0,t:C)}+\frac{C_3}{2}\int_0^t\left\|\frac{\partial\hat\gamma}{\partial z}\right\|^2_C\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial s}\right\|^2_{L^2(0,1)}ds,$$

$$\left|2C_3\int_0^t\int_\Omega \frac{1}{\hat\gamma}\frac{\partial^2\hat\gamma}{\partial z\partial s}v_{z_n}\frac{\partial\bar\gamma_m}{\partial s}\ rdrdzds\right|$$
$$\leq C_3\left\|v_{z_n}\right\|^2_{L^\infty(0,t:L^2(\Omega,r))}\left\|\frac{\partial^2\hat\gamma}{\partial z\partial t}\right\|^2_{L^2(0,t:C)}+\frac{C_3}{2\delta}\int_0^t\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial s}\right\|^2_{L^2(0,1)}ds.$$

The estimates of the seventh, eighth and the ninth term are standard:

$$2C_3\left|\int_0^t\int_\Omega \frac{1}{\hat\gamma}\frac{\partial\hat\gamma}{\partial z}\frac{\partial v_{z_n}}{\partial s}\frac{\partial\bar\gamma_m}{\partial s}rdrdzds\right|$$
$$\leq \epsilon_2\int_0^t\left\|\frac{\partial v_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)}ds+\frac{C_3^2}{\delta\epsilon_2}\int_0^t\left\|\frac{\partial\hat\gamma}{\partial z}\right\|^2_C\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial s}\right\|^2_{L^2(0,1)}ds,$$

$$\left|C_2\int_0^t\int_0^1\frac{1}{\hat\gamma}\left|\frac{\partial\bar\gamma_m}{\partial s}\right|^2\ dzds\right|\leq C_2\int_0^t\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial s}\right\|^2_{L^2(0,1)}ds.$$

$$\left|C_2\int_0^1\frac{1}{\hat\gamma}\frac{\partial\bar\gamma_m}{\partial t}\bar\gamma_m\ dz\right|\leq\frac{C_3}{4}\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial t}\right\|^2_{L^2(0,1)}+\frac{C_2^2}{C_3}\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^\infty(0,t:L^2(0,1))}.$$

The next term is a term with the initial data:

$$\left|C_2\int_0^1\frac{1}{\hat\gamma(z,0)}F_1(z,0)\bar\gamma_m(z,0)\ dz\right|\leq\frac{C_2}{2\delta}\left\|F_1(0)\right\|^2_{L^2(0,1)}+\frac{C_2}{2}\left\|\frac{1}{\hat\gamma(0)}\bar\gamma_m(0)\right\|^2_{L^2(0,1)}$$
$$\leq \hat K(C_2,1/\delta)\left((\gamma_1'(0)-\gamma_0'(0))^2+\gamma_0'(0)^2+\|\bar\gamma^0\|^2_{L^2(0,1)}\right),$$

where $\hat K>0$ is a constant depending on C_2 and $1/\delta$.

The estimates of the 11-th through the 13-th term make use of the L^∞ norms of v_{z_n} and $\bar\gamma_m$:

$$\left|2C_2\int_0^t\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial z}\frac{\partial\hat\gamma}{\partial s}\int_0^1 v_{z_n}\ rdr\bar\gamma_m\ dzds\right|$$
$$\leq\frac{C_2}{\delta^2}\left\|\frac{\partial\hat\gamma}{\partial t}\right\|^2_{L^2(0,t:C)}\left\|v_{z_n}\right\|^2_{L^\infty(0,t:L^2(\Omega,r))}+\frac{1}{\delta}\left\|\frac{\partial\hat\gamma}{\partial z}\right\|^2_{L^2(0,t:C)}\frac{C_2}{2}\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^\infty(0,t:L^2(0,1))},$$

$$\left|2C_2\int_0^t\int_\Omega \frac{1}{\hat\gamma}\frac{\partial^2\hat\gamma}{\partial z\partial s}v_{z_n}\bar\gamma_m\ rdrdzds\right|$$
$$\leq C_2\left\|\frac{\partial^2\hat\gamma}{\partial z\partial t}\right\|^2_{L^2(0,t:C)}\|v_{z_n}\|^2_{L^\infty(0,t:L^2(\Omega,r))}+\frac{C_2 t}{2\delta}\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^\infty(0,t:L^2(0,1))},$$

$$\left|2C_2\int_0^t\int_\Omega \frac{1}{\hat\gamma}\frac{\partial\hat\gamma}{\partial z}\frac{\partial v_{z_n}}{\partial s}\bar\gamma_m\ rdrdzds\right|$$
$$\leq \frac{C_2}{2\delta^2\epsilon_3}\left\|\frac{\partial\hat\gamma}{\partial z}\right\|^2_{L^2(0,t:C)}ds\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^\infty(0,t:L^2(0,1))}+\epsilon_3\int_0^t\left\|\frac{\partial v_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)}ds.$$

The estimate of the 14-th term can be obtained as follows:

$$\left|\int_0^t\int_\Omega F_2\frac{\partial v_{z_n}}{\partial s}\ rdrdzds\right|\leq\frac{1}{8\epsilon_4}\int_0^t F_2^2(t)dt+\epsilon_4\int_0^t\left\|\frac{\partial v_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)}ds.$$

In order to estimate the next two terms, we need to take into account the assumption of higher regularity of the boundary data, namely, $\gamma_0,\gamma_1\in H^2(0,T)$. We obtain the following estimates:

$$\left|C_3\int_0^t\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial s}F_1\frac{\partial\bar\gamma_m}{\partial t}dzds\right|\leq\frac{C_3}{2\delta}\int_0^t\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial s}\right\|^2_{L^2(0,1)}ds$$
$$+\frac{C_3}{2\delta^2}\sup_{0\leq s\leq t}\left(\frac{2}{3}|\gamma_1'(s)-\gamma_0'(s)|^2+2|\gamma_0'(s)|^2\right)\left\|\frac{\partial\hat\gamma}{\partial t}\right\|^2_{L^2(0,t:C)},$$

and

$$\left|C_3\int_0^t\int_0^1\frac{1}{\hat\gamma}\frac{\partial F_1}{\partial s}\frac{\partial\bar\gamma_m}{\partial t}dzds\right|\leq\frac{C_3}{2\delta}\int_0^t\left\|\frac{1}{\sqrt{\hat\gamma}}\frac{\partial\bar\gamma_m}{\partial s}\right\|^2_{L^2(0,1)}ds$$
$$+\frac{C_3}{3\delta}\|\gamma_L''-\gamma_0''\|^2_{L^2(0,T)}+\frac{C_3}{\delta}\|\gamma_0''\|^2_{L^2(0,T)}.$$

Similarly, the last two terms can be estimated as:

$$\left|C_2\int_0^t\int_0^1\frac{1}{\hat\gamma^2}\frac{\partial\hat\gamma}{\partial s}F_1\bar\gamma_m dzds\right|\leq\frac{C_2}{2\delta}\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^\infty(0,t:L^2(0,1))}$$
$$+\frac{C_2}{2\delta^2}\sup_{0\leq s\leq t}\left(\frac{2}{3}|\gamma_1'(s)-\gamma_0'(s)|^2+2|\gamma_0'(s)|^2\right)\left\|\frac{\partial\hat\gamma}{\partial t}\right\|^2_{L^2(0,t:C)},$$

and

$$\left|C_2\int_0^t\int_0^1\frac{1}{\hat\gamma}\frac{\partial F_1}{\partial s}\bar\gamma_m dzds\right|\leq\frac{C_2}{2\delta}\left\|\frac{1}{\sqrt{\hat\gamma}}\bar\gamma_m\right\|^2_{L^\infty(0,t:L^2(0,1))}$$
$$+\frac{C_2}{3\delta}\|\gamma_L''-\gamma_0''\|^2_{L^2(0,T)}+\frac{C_2}{\delta}\|\gamma_0''\|^2_{L^2(0,T)}.$$

By combining the estimates above and by choosing $\sum_{i=1}^4 \epsilon_i \leq \frac{3}{4}$, which is relevant only in the coefficient in front of $\int_0^t \left\|\frac{\partial v_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)} ds$ on the left hand-side in the inequality below, we obtain

$$
\begin{aligned}
&\frac{C_3}{4}\left\|\frac{1}{\sqrt{\hat{\gamma}}}\frac{\partial\bar{\gamma}_m}{\partial t}\right\|^2_{L^2(0,1)} + \frac{C_1}{2}\left\|\frac{1}{\hat{\gamma}}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)} + \frac{1}{4}\int_0^t \left\|\frac{\partial v_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)} ds \\
&\leq \frac{C_3}{4}\int_0^t \left[\frac{2}{\delta}\left\|\frac{\partial\hat{\gamma}}{\partial s}\right\|_C + 2\left\|\frac{\partial\hat{\gamma}}{\partial z}\right\|^2_C + \frac{4C_3}{\delta\epsilon_2}\left\|\frac{\partial\hat{\gamma}}{\partial z}\right\|^2_C\right]\left\|\frac{1}{\sqrt{\hat{\gamma}}}\frac{\partial\bar{\gamma}_m}{\partial s}\right\|^2_{L^2(0,1)} ds \\
&+ \frac{C_1}{2}\int_0^t \left[\frac{2}{\delta}\left\|\frac{\partial\hat{\gamma}}{\partial s}\right\|_C + \frac{1}{2\epsilon_1 C_1}\left\|\frac{\partial\hat{\gamma}}{\partial s}\right\|^2_C\right]\left\|\frac{1}{\hat{\gamma}}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)} ds \\
&+ \left[\frac{3C_3}{2\delta} + \frac{C_2}{C_3}\right]\left\|\sqrt{\hat{\gamma}}\frac{\partial\bar{\gamma}_m}{\partial t}\right\|^2_{L^2(0,T:L^2(0,1))} \\
&+ \left(\frac{C_2 + C_3\delta}{\delta^2}\left\|\frac{\partial\hat{\gamma}}{\partial t}\right\|^2_{L^2(0,T:C)} + (C_3 + C_2)\left\|\frac{\partial^2\hat{\gamma}}{\partial z\partial t}\right\|^2_{L^2(0,T:C)}\right)\|v_{z_n}\|^2_{L^\infty(0,T:L^2(\Omega,r))} \\
&+ C_2\left[\frac{C_2}{C_3} + \frac{T+2}{2\delta} + \left(\frac{1}{2\delta} + \frac{1}{2\delta^2\epsilon_3}\right)\left\|\frac{\partial\hat{\gamma}}{\partial z}\right\|^2_{L^2(0,T:C)}\right]\left\|\frac{1}{\sqrt{\hat{\gamma}}}\bar{\gamma}_m\right\|^2_{L^\infty(0,T:L^2(0,1))} \\
&+ K_1(C_2, C_3, 1/\delta, \|\hat{\gamma}\|^2_{H^1(0,T:C)}) \sup_{0\leq s\leq T}\left(|\gamma_1'(s) - \gamma_0'(s)|^2 + |\gamma_0'(s)|^2\right) \\
&+ K_2(C_2, C_3, 1/\delta)\left(\|\gamma_1 - \gamma_0\|^2_{H^2(0,T)} + \|\gamma_0\|^2_{H^2(0,T)}\right) \\
&+ K_3(C_1, C_2, C_3, 1/\delta, \|\hat{\gamma}(0)\|^2_{C^1[0,1]})\left(\|\bar{\gamma}^0\|^2_{L^2(0,1)} + \|v_z^0\|^2_{H^1_{0,0}(\Omega,r)}\right),
\end{aligned}
\tag{5.5}
$$

where K_1, K_2 and K_3 are positive constants which depend on the quantities listed in the corresponding parentheses.

This inequality is of the form

$$X(t) + \int_0^t A(s)ds \leq \int_0^t B(s)X(s)ds + D$$

where

$$
\begin{aligned}
X(t) &:= \frac{C_3}{4}\left\|\frac{1}{\sqrt{\hat{\gamma}}}\frac{\partial\bar{\gamma}_m}{\partial t}\right\|^2_{L^2(0,1)} + \frac{C_1}{2}\left\|\frac{1}{\hat{\gamma}}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)}, \\
A(t) &:= \left\|\frac{\partial v_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)},
\end{aligned}
$$

and

$$B(s) \text{ depends on } \left\|\frac{\partial\hat{\gamma}}{\partial t}\right\|_C, \left\|\frac{\partial\hat{\gamma}}{\partial z}\right\|^2_C, 1/\delta, 1/C_1, \text{ and } C_3,$$

and D consists of all the terms appearing in (5.5) in rows 4 through 9. We can now apply the following form of Gronwall's inequality to obtain the desired estimate:

Lemma 5.2 *(Gronwall, [12]) Let $X(t)$ be a nonnegative, summable function on $[0,T]$ which satisfies for a.e. t the integral inequality*

$$X(t) + \int_0^t A(s)ds \leq \int_0^t B(s)X(s)ds + D$$

for $A(t), B(t), D \geq 0$ for all t. Then

$$X(t) + \int_0^t A(s)ds \leq D\left(1 + e^{\int_0^t B(s)ds}\int_0^t B(s)ds\right)$$

Before we apply this form of Gronwall's inequality, first notice that the norms of $\bar{\gamma}_m$ and v_{z_n}, appearing in the term denoted by D, are all bounded by the initial and boundary data via the first energy estimate presented in Theorem 4.3. Thus, by using the energy estimate presented in Theorem 4.3 and by employing the Gronwall's inequality presented in Lemma 5.2, we conclude that there exists a constant $C > 0$ depending on

$$\|\hat{\gamma}\|_{H^1(0,T:C^1[0,1])}, 1/\delta, C_1, C_2, C_3 \text{ and } T,$$

such that

$$\begin{aligned}&\text{ess sup}_{0\leq t\leq T}\left\{\left\|\frac{1}{\sqrt{\hat{\gamma}}}\frac{\partial\bar{\gamma}_m}{\partial t}\right\|^2_{L^2(\Omega,r)} + \left\|\frac{1}{\hat{\gamma}}\frac{\partial v_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)}\right\} + \left\|\frac{\partial v_{z_n}}{\partial t}\right\|^2_{L^2(0,T:L^2(\Omega,r))}\\ \leq &C\left(\|\bar{\gamma}^0\|^2_{L^2(0,1)} + \|v_z^0\|^2_{H^1_{0,0}(\Omega,r)} + \|\gamma_1-\gamma_0\|^2_{H^2(0,T)} + \|\gamma_0\|^2_{H^2(0,T)}\right)\end{aligned}$$

Passing to the limit as $m \to \infty$ and $n \to \infty$ we recover the estimate from the statement of the theorem. ■

We now show that $\dfrac{\partial\bar{\gamma}}{\partial z}$ and $\dfrac{\partial^2\bar{\gamma}}{\partial z\partial t}$ are in $L^2(0,T:L^2(0,1))$.

Theorem 5.3 **(Improved Regularity: Part II)** *Assume, in addition to the assumptions of Theorem 5.1, that the initial data $\bar{\gamma}^0 \in H^1(0,1)$. Then, the function $\bar{\gamma}$, which corresponds to a weak solution to problem* (2.2)-(2.6)*, satisfies*

$$\frac{\partial\bar{\gamma}}{\partial z} \in L^2(0,T:L^2(0,1)) \text{ and } \frac{\partial^2\bar{\gamma}}{\partial z\partial t} \in L^2(0,T:L^2(0,1)).$$

Moreover, the following estimate holds:

$$\begin{aligned}&\left\|\frac{\partial\bar{\gamma}}{\partial z}\right\|_{L^2(0,1)} + \left\|\frac{\partial^2\bar{\gamma}}{\partial z\partial t}\right\|_{L^2(0,T:L^2(\Omega,r))} \leq\\ &C\left(\|\bar{\gamma}^0\|_{H^1(0,1)} + \|v_z^0\|_{H^1_{0,0}(\Omega,r)} + \|\gamma_1-\gamma_0\|^2_{H^2(0,T)} + \|\gamma_0\|^2_{H^2(0,T)}\right).\end{aligned}$$

This implies that, in fact,

$$\frac{\partial\bar{\gamma}}{\partial z} \in L^\infty(0,T:L^2(0,1)), \quad \frac{\partial^2}{\partial z^2}\int_0^1 v_z r dr \in L^2(0,T:L^2(0,1)),$$

$$\frac{\partial^2\bar{\gamma}}{\partial t^2} \in H^{-1}(0,T:L^2(0,1)) \text{ and } \Delta_r v_z \in L^2(0,T:L^2(\Omega,r)).$$

Proof: The proof is based on the following idea. We will use the weak form of the momentum equation (4.4) to estimate $\frac{\partial\bar{\gamma}}{\partial z}$ and $\frac{\partial^2\bar{\gamma}}{\partial z\partial t}$. In order to obtain the desired estimate, we would like to substitute the test function w_k in the weak form of the momentum equation (4.4) by $\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}$ to get the L^2-norm of $\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}$ on the right hand-side of the equation. Substituting w_k by $\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}$ is, however, not possible since function $\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}$ lives in a different space from the space of test functions w_k. This problem can be rectified by considering the function

$$(1-r)\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t} = (1-r)\sum_{k=1}^m \dot{d}_k^m(t)\frac{\partial\phi_k(z)}{\partial z} = \sum_{k=1}^m \dot{d}_k^m(t)\underbrace{(1-r)\frac{\partial\phi_k(z)}{\partial z}}_{w_k(r,z)},$$

and taking

$$w_k(r,z) = (1-r)\frac{\partial\phi_k(z)}{\partial z} \in C^1_{0,0}(\Omega,r). \tag{5.6}$$

Now notice that, without loss of generality, we could have used the space $C^1_{0,0}$ in the definition of the Galerkin approximation for the velocity, instead of the space $C^\infty_{0,0}$. Thus, everything obtained so far holds assuming $w_k \in C^1_{0,0}$. This relaxed choice of the space for w_k is now important to obtain improved regularity.

With this observation we can now proceed by substituting w_k in (4.4) with (5.6) and by multiplying equation (4.4) by $\dot{d}_k^m(t)$ and summing over $k = 1,...m$ to obtain

$$\begin{aligned}
&\int_0^t\int_\Omega \frac{\partial v_z}{\partial t}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\,(1-r)rdrdzds - C_1\int_0^t\int_\Omega \frac{1}{\hat{\gamma}^2}\frac{\partial v_z}{\partial r}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\,rdrdzds\\
&-\int_0^t\int_\Omega \frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial v_z}{\partial r}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\,(1-r)r^2drdzds = -C_2\int_0^t\int_\Omega \frac{\partial\bar{\gamma}_m}{\partial z}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\,(1-r)rdrdzds\\
&-C_3\int_0^t\int_\Omega \left|\frac{\partial\bar{\gamma}_m}{\partial z\partial t}\right|^2\,(1-r)rdrdzds - \int_0^t\int_\Omega F_2\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\,(1-r)rdrdzds.
\end{aligned}$$

This is an equation that will be used to obtain the desired estimate bounding the z-derivatives of $\bar{\gamma}_m$. We multiply the above equation by C_3 and rewrite it slightly to obtain

$$\frac{C_2C_3}{12}\int_0^1\left|\frac{\partial\bar{\gamma}_m}{\partial z}\right|^2\ dz+\frac{C_3^2}{6}\int_0^t\int_0^1\left|\frac{\partial\bar{\gamma}_m}{\partial z\partial t}\right|^2\ dzds=\frac{C_2C_3}{12}\int_0^1\left|\frac{\partial\bar{\gamma}_m(z,0)}{\partial z}\right|^2\ dz$$
$$-C_3\int_0^t\int_\Omega\frac{\partial v_z}{\partial t}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ (1-r)rdrdzds+C_1C_3\int_0^t\int_\Omega\frac{1}{\hat{\gamma}^2}\frac{\partial v_z}{\partial r}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ rdrdzds$$
$$+C_3\int_0^t\int_\Omega\frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial v_z}{\partial r}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ (1-r)r^2drdzds-C_3\int_0^t\int_\Omega F_2\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ (1-r)rdrdzds.$$

We can now estimate the terms on the right-hand side to obtain the desired estimate. We proceed as follows. The second and third terms on the right hand side can be estimated as:

$$\left|C_3\int_0^t\int_\Omega\frac{\partial v_z}{\partial t}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ (1-r)rdrdzds\right|$$
$$\leq\frac{1}{2}\int_0^t\int_\Omega\left|\frac{\partial v_z}{\partial t}\right|^2rdrdzds+\frac{C_3^2}{24}\int_0^t\int_0^1\left|\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\right|^2dzds,$$

$$\left|C_1C_3\int_0^t\int_\Omega\frac{1}{\hat{\gamma}^2}\frac{\partial v_z}{\partial r}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ rdrdzds\right|$$
$$\leq\frac{3C_1^2}{\delta^2}\int_0^t\int_\Omega\frac{1}{\hat{\gamma}^2}\left|\frac{\partial v_z}{\partial r}\right|^2rdrdzds+\frac{C_3^2}{24}\int_0^t\int_0^1\left|\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\right|^2dzds.$$

Similarly, the fourth and the fifth terms are estimated as:

$$\left|C_3\int_0^t\int_\Omega\frac{1}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial v_z}{\partial r}\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ (1-r)r^2drdzds\right|$$
$$\leq\frac{1}{2}\int_0^t\left\|\frac{\partial\hat{\gamma}}{\partial t}\right\|_C^2\int_\Omega\frac{1}{\hat{\gamma}^2}\left|\frac{\partial v_z}{\partial r}\right|^2rdrdzds+\frac{C_3^2}{120}\int_0^t\int_0^1\left|\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\right|^2dzds,$$

$$\left|C_3\int_0^t\int_\Omega F_2\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\ (1-r)rdrdzds\right|$$
$$\leq\frac{1}{4}\int_0^t\int_0^1|F_2|^2\,dzds+\frac{C_3^2}{24}\int_0^t\int_0^1\left|\frac{\partial^2\bar{\gamma}_m}{\partial z\partial t}\right|^2dzds.$$

This leads to

$$\frac{C_2C_3}{12}\int_0^1\left|\frac{\partial\bar{\gamma}_m}{\partial z}\right|^2\ dz+\frac{C_3^2}{30}\int_0^t\int_0^1\left|\frac{\partial\bar{\gamma}_m}{\partial z\partial t}\right|^2\ dzds\leq\frac{C_2C_3}{12}\int_0^1\left|\frac{\partial\bar{\gamma}^0}{\partial z}\right|^2\ dz$$
$$\frac{1}{2}\left\|\frac{\partial v_z}{\partial t}\right\|^2_{L^2(0,T:L^2)}+\left[\frac{3TC_1^2}{\delta^2}+\frac{1}{2}\left\|\frac{\partial\hat{\gamma}}{\partial t}\right\|^2_{L^2(0,T:C)}\right]\left\|\frac{1}{\hat{\gamma}}\frac{\partial v_z}{\partial r}\right\|^2_{L^\infty(0,T:L^2)}+\frac{1}{4}\int_0^t|F_2(s)|^2ds.$$

By combining the energy estimate stated in Theorem 4.3, and the improved regularity estimate stated in Theorem 5.1 we see that there exits a constant $C > 0$ such that

$$\left\| \frac{\partial \bar{\gamma}_m}{\partial z} \right\|^2_{L^2(0,1)} + \left\| \frac{\partial \bar{\gamma}_m}{\partial z \partial t} \right\|^2_{L^2(0,T:L^2(\Omega,r))} \leq$$
$$C \left(\|\bar{\gamma}^0\|_{H^1(0,1)} + \|v_z^0\|_{H^1_{0,0}(\Omega,r)} + \|\gamma_1 - \gamma_0\|^2_{H^2(0,T)} + \|\gamma_0\|^2_{H^2(0,T)} \right).$$

By passing to the limit as $m \to \infty$ we recover the result of Theorem 5.3.

■

With this regularity, our system of partial differential equations is satisfied almost everywhere in $\Omega \times (0, T)$, which means that, under the assumptions of higher regularity of the initial and boundary data as listed in Theorem 5.3, we have shown the existence of a *mild solution* to the original problem.

6 Conclusions

We have established existence and uniqueness of a weak solution to a three-dimensional axially symmetric Biot-like problem modeling blood flow. Assuming that initial data for the displacement and velocity are in L^2, and that the boundary data for the displacement are in H^1, we proved the existence of a weak solution to problem (1.3)-(1.6). The weak solution is less regular than a standard parabolic solution due to the hyperbolic-degenerate parabolic nature of the problem. We further showed that, if the data is more regular, our solution has higher regularity, satisfying (1.3)-(1.4) almost everywhere in $\Omega \times (0, T)$.

It is interesting to notice that the main regularization in this problem comes from the structure viscoelasticity which governs the time evolution of the structure motion and provides estimates for the regularity in the axial direction of the average (axial) velocity and of the wave fronts propagating in the structure. Thus, vessel wall viscoelasticity plays a crucial role in smoothing out the sharp fronts generated by the steep pressure pulse emanating from the heart.

It is worth mentioning that the numerical simulations involving the linear model (1.3)-(1.6) linearized around $\hat{\gamma} = R$ and with time-periodic inlet/outlet boundary data for γ corresponding to a physiologically reasonable pressure pulse, show excellent agreement with the experimental measurements of the three-dimensional axially symmetric flow in a compliant tube [9].

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