

UNIQUE EQUILIBRIUM STATES FOR BONATTI–VIANA DIFFEOMORPHISMS

VAUGHN CLIMENHAGA, TODD FISHER, AND DANIEL J. THOMPSON

ABSTRACT. We show that the robustly transitive diffeomorphisms constructed by Bonatti and Viana have unique equilibrium states for natural classes of potentials. In particular, we characterize the SRB measure as the unique equilibrium state for a suitable geometric potential. The techniques developed are applicable to a wide class of DA diffeomorphisms, and persist under C^1 perturbations of the map. These results are an application of general machinery developed by the first and last named authors.

1. INTRODUCTION AND STATEMENT OF RESULTS

An *equilibrium state* for a diffeomorphism $f: M \rightarrow M$ and a potential $\varphi: M \rightarrow \mathbb{R}$ is an invariant Borel probability measure that maximizes the quantity $h_\mu(f) + \int \varphi d\mu$. Results on existence and uniqueness of equilibrium states have a long history [5, 20, 23, 9, 39, 18, 31, 46], and are one of the main goals in thermodynamic formalism. Such results are a powerful tool to understand the orbit structure and global statistical properties of dynamical systems, and often lead to further applications, including large deviations principles, central limit theorems, and knowledge of dynamical zeta functions [30, 45].

The benchmark result of this type is that there is a unique equilibrium state μ when (M, f) is uniformly hyperbolic, mixing, and φ is Hölder continuous. When φ is the *geometric potential* $\varphi(x) = -\log \det(Df|_{E^u(x)})$, this unique equilibrium state is the SRB measure [41, 5, 37]. Extending this type of result beyond uniform hyperbolicity is a major challenge in the field. The first and third authors have developed techniques to establish existence and uniqueness of equilibrium states in the presence of non-uniform versions of specification and expansivity [19]. The purpose of this paper is to show how these results can be applied to higher dimensional smooth systems with weak forms of hyperbolicity, where alternative approaches based on symbolic

Date: October 22, 2017.

V.C. is supported by NSF grant DMS-1554794. T.F. is supported by Simons Foundation grant # 239708. D.T. is supported by NSF grant DMS-1461163.

dynamics or transfer operators appear to meet with fundamental difficulties. While thermodynamic formalism for one-dimensional systems is well developed, and there have been major recent breakthroughs in dimension two by Buzzi, Crovisier, and Sarig [40, 11], the higher-dimensional case remains poorly understood.

We focus on the class of Bonatti–Viana diffeomorphisms [4]; these are robustly transitive, derived from Anosov (DA), diffeomorphisms of \mathbb{T}^4 . These are the model examples of robustly transitive DA systems with a dominated splitting that are not partially hyperbolic. This setting demonstrates the flexibility of our methods while having the advantage of being concrete. To the best of our knowledge, no other techniques for uniqueness are available for the class of Bonatti–Viana diffeomorphisms, or indeed any natural class of smooth systems on \mathbb{T}^4 beyond uniform hyperbolicity.

The Bonatti–Viana construction [4, 12] is a C^0 perturbation of a 4-dimensional toral automorphism f_A with a hyperbolic splitting $E^s \oplus E^u$, where $\dim E^s = \dim E^u = 2$. The perturbation has a dominated splitting $E^{cs} \oplus E^{cu}$ and can be characterized by two parameters ρ and λ : the parameter $\rho > 0$ is the size of the balls $B(q, \rho) \cup B(q', \rho)$ inside which the perturbation takes place, where q, q' are fixed points; the parameter $\lambda > 1$ is the maximum of expansion in the centre-stable and expansion in backwards time in the centre-unstable. The construction can be carried out with both ρ and $\log \lambda$ arbitrarily small if required.

For fixed $\lambda > 1$ and $\rho > 0$, we write $f_{BV} \in \mathcal{F}_{\lambda, \rho}$ for a diffeomorphism provided by the Bonatti–Viana construction for which these parameters are bounded above by these values of λ and ρ . Our results give quantitative criteria for existence and uniqueness of the equilibrium state in terms of a function Φ which depends on the size of the perturbation from the original Anosov map (via the parameters ρ and λ), the norm and variation of the potential, and the tail entropy of the system.

Theorem A. *Let $f_{BV} \in \mathcal{F}_{\lambda, \rho}$ be a diffeomorphism obtained by the Bonatti–Viana construction and g be a C^1 perturbation of f_{BV} . Let $\varphi: \mathbb{T}^4 \rightarrow \mathbb{R}$ be a Hölder continuous potential function. There is a function $\Phi = \Phi(\varphi; g)$, given explicitly at (4.6), such that*

- (1) $\lim_{\lambda \rightarrow 1, \rho \rightarrow 0} \sup\{\Phi(\varphi; f) : f \in \mathcal{F}_{\lambda, \rho}\} = \max\{\varphi(q), \varphi(q')\} < P(\varphi; f_A)$;
- (2) $\Phi(\varphi; g)$ varies continuously under C^0 perturbation of the potential function and C^1 perturbation of the map;

and if $\Phi(\varphi; g) < P(\varphi; g)$, then $(\mathbb{T}^4, g, \varphi)$ has a unique equilibrium state.

A more precise statement of this result, including the definition of Φ , is given as Theorem 4.1. In §5, by analyzing the function Φ and the

topological pressure $P(\varphi; g)$, we give two corollaries of this result: in Corollary 5.1, we show that for a fixed diffeomorphism g which is a C^1 perturbation of f_{BV} , every Hölder continuous φ satisfying a bounded range condition has a unique equilibrium state; in Corollary 5.2, we show that for a fixed Hölder continuous potential then there is a unique equilibrium state with respect to f_{BV} as long as the parameters $\rho, \log \lambda$ in the Bonatti-Viana construction are sufficiently small.

We also consider potential functions φ which are scalar multiples of the geometric potential $\varphi^{\text{geo}}(x) = -\log \det(Df|_{E^{cu}}(x))$. We obtain the following results.

Theorem B. *Let $f_{BV} \in \mathcal{F}_{\lambda, \rho}$ with $\log \lambda, \rho$ sufficiently small. Then for every C^2 diffeomorphism g which is a sufficiently small C^1 perturbation of f_{BV} , the following are true.*

- $t = 1$ is the unique root of the function $t \mapsto P(t\varphi_g^{\text{geo}}; g)$.
- There is an $\varepsilon > 0$ such that $t\varphi_g^{\text{geo}}$ has a unique equilibrium state μ_t for each $t \in (-\varepsilon, 1 + \varepsilon)$.
- μ_1 is the unique SRB measure for g .

Our results are proved using general machinery developed by the first and last named authors [19]. The idea is to find a ‘good’ collection of orbit segments on which the map has uniform expansion, contraction, and mixing properties, and demonstrating that this collection is large in the sense that any orbit segment can be *decomposed* into ‘good’ and ‘bad’ parts in such a way that the collection of ‘bad’ orbit segments has smaller topological pressure than the entire system.

The diffeomorphisms we consider are not expansive. In particular, a C^1 perturbation of a diffeomorphism f_{BV} may not even be asymptotically h-expansive, and thus may have positive tail entropy [12]. We handle this by showing that any measure with large enough free energy is *almost expansive* (Definition 2.3), so the failure of expansivity does not affect equilibrium states. A significant technical point in our approach is that we carry out all our estimates coarsely at a definite fixed scale which is too large to ‘see’ the bad dynamics that may occur at small scales or infinitesimally. We control the smaller scales indirectly using bounds on the tail entropy. The need to be precise and careful about scales leads to substantial technicalities in our arguments.

We now review the relevant results in the literature. For systems with a dominated splitting, there are some known results on uniqueness of the measure of maximal entropy although these mostly require partial hyperbolicity [2, 43, 36], and the case of equilibrium states for $\varphi \neq 0$ have been largely unexplored. For the Bonatti–Viana examples, the existence of a unique MME was obtained in [12], using a technique

that is not suited to generalization to equilibrium states. Existence of equilibrium states for partially hyperbolic horseshoes was studied by Leplaideur, Oliveira, and Rios [26], but they do not deal with uniqueness. Results for uniqueness of equilibrium states for frame flows have been obtained recently by Spatzier and Visscher [42]. Other references which apply in higher dimensional settings include [14, 46]. In particular, Pesin, Senti and Zhang [46] have used tower techniques to develop thermodynamic formalism for the Katok map, which is a non-uniformly hyperbolic DA map of the 2-torus.

The theory of SRB measures has received much more attention. The fact that there is a unique SRB measure for the examples we study follows from [4]. The connection between SRB measures and equilibrium states is given by the Ledrappier–Young formula and the Margulis–Ruelle inequality. However, even when there is known to be a unique SRB measure, the characterization of the SRB as an equilibrium state of a continuous potential function requires a non-trivial proof because the number of positive Lyapunov exponents can be different for different measures. For diffeomorphisms with a dominated splitting, Carvalho [15] has showed that the SRB measure for a DA system obtained along an arc of C^∞ diffeomorphisms is an equilibrium state. Along a C^r arc, her result only applied at first bifurcation. To the best of our knowledge, our results in §7 are the first that characterize the SRB measure as a *unique* equilibrium state for a class of diffeomorphisms with a dominated splitting beyond uniform hyperbolicity.

The techniques introduced in this paper are robust and apply for other DA systems. In [17], we use the tools and ideas introduced in this paper to study the partially hyperbolic Mañé family of diffeomorphisms [27]. This family is significantly easier to study than the Bonatti–Viana family, and we are able to derive stronger results in that setting. Almost Anosov diffeomorphisms, Katok maps, and the Shub class of robustly transitive diffeomorphisms [22] are other classes of DA systems where these techniques can be explored.

In §2, we give the necessary background material from [19] on thermodynamic formalism. In §3, we prove general pressure estimates for C^0 -perturbations of Anosov systems. In §4, we provide details of the Bonatti–Viana construction, and state a more precise version of Theorem A. In §5, we prove corollaries of Theorem A. In §6, we prove our main theorem. In §7, we prove Theorem B on SRB measures. In §8, we provide proofs for a few technical lemmas.

2. BACKGROUND

In this section, we state definitions and results that we will need throughout the paper.

2.1. Pressure. Let X be a compact metric space and $f: X \rightarrow X$ be a continuous map. Henceforth, we will identify $X \times \mathbb{N}$ with the space of finite orbit segments for a map f via the correspondence

$$(2.1) \quad (x, n) \longleftrightarrow (x, f(x), \dots, f^{n-1}(x)).$$

For a continuous potential function $\varphi: X \rightarrow \mathbb{R}$ we write

$$S_n \varphi(x) = S_n^f \varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k x)$$

for the ergodic sum along an orbit segment, and given $\eta > 0$, we write

$$\text{Var}(\varphi, \eta) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in X, d(x, y) < \eta\}.$$

Given $n \in \mathbb{N}$ and $x, y \in X$, we write

$$d_n(x, y) = \max\{d(f^k x, f^k y) : 0 \leq k < n\}.$$

Given $x \in X$, $\varepsilon > 0$, and $n \in \mathbb{N}$, the *Bowen ball of order n with center x and radius ε* is

$$B_n(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}.$$

We say that $E \subset X$ is (n, ε) -separated if $d_n(x, y) \geq \varepsilon$ for all $x, y \in E$.

We will need to consider the *pressure of a collection of orbit segments*. More precisely, we interpret $\mathcal{D} \subset X \times \mathbb{N}$ as a collection of finite orbit segments, and write $\mathcal{D}_n = \{x \in X : (x, n) \in \mathcal{D}\}$ for the set of initial points of orbits of length n in \mathcal{D} . Then we consider the partition sum

$$\Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, \varepsilon; f) = \sup \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : E \subset \mathcal{D}_n \text{ is } (n, \varepsilon)\text{-separated} \right\}.$$

When there is no confusion in the map we will sometimes omit the dependence on f from our notation. We will also sometimes require a partition sum Λ_n^{span} defined with (n, ε) -spanning sets. Given $Y \subset X$, $n \in \mathbb{N}$, and $\delta > 0$, we say that $E \subset Y$ is an (n, δ) -spanning set for Y if $\bigcup_{x \in E} \overline{B_n(x, \delta)} \supset Y$. Write

$$\Lambda_n^{\text{span}}(\mathcal{D}, \varphi, \delta; f) = \inf \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : E \subset \mathcal{D}_n \text{ is } (n, \delta)\text{-spanning} \right\}.$$

We will use the following basic result relating Λ_n^{sep} and Λ_n^{span} , which is proved in §8.

Lemma 2.1. *For any $\mathcal{D} \subset X \times \mathbb{N}$, $\varphi: X \rightarrow \mathbb{R}$, and $\delta > 0$, we have*

$$\begin{aligned}\Lambda_n^{\text{span}}(\mathcal{D}, \varphi, \delta) &\leq \Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, \delta), \\ \Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, 2\delta) &\leq e^{n \text{Var}(\varphi, \delta)} \Lambda_n^{\text{span}}(\mathcal{D}, \varphi, \delta).\end{aligned}$$

The pressure of φ on \mathcal{D} at scale ε is

$$P(\mathcal{D}, \varphi, \varepsilon; f) = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, \varepsilon),$$

and the pressure of φ on \mathcal{D}

$$P(\mathcal{D}, \varphi; f) = \lim_{\varepsilon \rightarrow 0} P(\mathcal{D}, \varphi, \varepsilon; f).$$

The above definition is a non-stationary version of the usual upper capacity pressure [32]. For a set $Z \subset X$, we let $P(Z, \varphi, \varepsilon; f) := P(Z \times \mathbb{N}, \varphi, \varepsilon; f)$, and thus $P(Z, \varphi; f)$ is the usual upper capacity pressure.

When $\varphi = 0$ the above definition gives the *entropy of \mathcal{D}* :

$$(2.2) \quad h(\mathcal{D}, \varepsilon; f) = h(\mathcal{D}, \varepsilon) := P(\mathcal{D}, 0, \varepsilon) \text{ and } h(\mathcal{D}) = \lim_{\varepsilon \rightarrow 0} h(\mathcal{D}, \varepsilon).$$

We let $\mathcal{M}(f)$ denote the set of f -invariant Borel probability measures and $\mathcal{M}_e(f)$ the set of ergodic f -invariant Borel probability measures. The variational principal for pressure [44, Theorem 9.10] states that if X is a compact metric space and f is continuous, then

$$P(\varphi; f) = \sup_{\mu \in \mathcal{M}(f)} \left\{ h_\mu(f) + \int \varphi d\mu \right\} = \sup_{\mu \in \mathcal{M}_e(f)} \left\{ h_\mu(f) + \int \varphi d\mu \right\}.$$

A measure achieving the supremum is an *equilibrium state*, and these are the objects whose existence and uniqueness we wish to study.

2.2. Expansivity and tail entropy. Given a homeomorphism $f: X \rightarrow X$ and $\varepsilon > 0$, consider for each $x \in X$ and $\varepsilon > 0$ the set

$$\Gamma_\varepsilon(x) := \{y \in X : d(f^k x, f^k y) < \varepsilon \text{ for all } k \in \mathbb{Z}\}$$

is the (*bi-infinite*) *Bowen ball of x of size ε* . Note that f is expansive if and only if there exists $\varepsilon > 0$ so that $\Gamma_\varepsilon(x) = \{x\}$ for all $x \in X$.

For systems that fail to be expansive, it is useful to consider the *tail entropy of f at scale $\varepsilon > 0$* is

$$(2.3) \quad h_f^*(\varepsilon) = \sup_{x \in X} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \Lambda_n^{\text{span}}(\Gamma_\varepsilon(x) \times \mathbb{N}, 0, \delta; f).$$

This quantity was introduced in [6]; equivalent definitions can also be formulated using open covers [29].

The map f is entropy-expansive if $h_f^*(\varepsilon) = 0$ for some $\varepsilon > 0$, and is asymptotically h -expansive if $h_f^*(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. See [8, 10] for

connections between these notions and the theory of symbolic extensions. An interesting result of [8] is that positive tail entropy rules out the existence of a principal symbolic extension, and thus symbolic dynamics fails in a strong way for such systems.

Given a collection $\mathcal{D} \subset X \times \mathbb{N}$ and scales $0 < \delta < \varepsilon$, the tail entropy allows us to control pressure at scale δ in terms of pressure at scale ε . The following is proved in §8.

Lemma 2.2. *Given any $\mathcal{D} \subset X \times \mathbb{N}$ and $0 < \delta < \varepsilon$, we have*

$$P(\mathcal{D}, \varphi, \delta; f) \leq P(\mathcal{D}, \varphi, \varepsilon; f) + h_f^*(\varepsilon) + \text{Var}(\varphi, \varepsilon) + \text{Var}(\varphi, \delta).$$

In particular, $P(\mathcal{D}, \varphi; f) \leq P(\mathcal{D}, \varphi, \varepsilon; f) + h_f^(\varepsilon) + \text{Var}(\varphi, \varepsilon)$.*

2.3. Obstructions to expansivity, specification, and regularity.

It was shown by Bowen [7] that (X, f, φ) has a unique equilibrium state whenever (X, f) has expansivity and specification, and φ is sufficiently regular. We require the results from [19], which give existence and uniqueness as long as ‘obstructions to specification and regularity’ and ‘obstructions to expansivity’ have smaller pressure than the whole system. We recall the necessary definitions, which can be found in [19].

2.3.1. *Expansivity.* We introduce the following quantity associated with the set of non-expansive points.

Definition 2.3. *For $f: X \rightarrow X$ the set of non-expansive points at scale ε is $\text{NE}(\varepsilon) := \{x \in X : \Gamma_\varepsilon(x) \neq \{x\}\}$. An f -invariant measure μ is almost expansive at scale ε if $\mu(\text{NE}(\varepsilon)) = 0$. Given a potential φ , the pressure of obstructions to expansivity at scale ε is*

$$\begin{aligned} P_{\text{exp}}^\perp(\varphi, \varepsilon) &= \sup_{\mu \in \mathcal{M}_e(f)} \left\{ h_\mu(f) + \int \varphi d\mu : \mu(\text{NE}(\varepsilon)) > 0 \right\} \\ &= \sup_{\mu \in \mathcal{M}_e(f)} \left\{ h_\mu(f) + \int \varphi d\mu : \mu(\text{NE}(\varepsilon)) = 1 \right\}. \end{aligned}$$

This is monotonic in ε , so we can define a scale-free quantity by

$$P_{\text{exp}}^\perp(\varphi) = \lim_{\varepsilon \rightarrow 0} P_{\text{exp}}^\perp(\varphi, \varepsilon).$$

2.3.2. *Specification.* We define specification for a collection of orbit segments.

Definition 2.4. *A collection of orbit segments $\mathcal{G} \subset X \times \mathbb{N}$ has specification at scale ε if there exists $\tau \in \mathbb{N}$ such that for every $\{(x_j, n_j) : 1 \leq j \leq k\} \subset \mathcal{G}$, there is a point x in*

$$\bigcap_{j=1}^k f^{-(m_j-1+\tau)} B_{n_j}(x_j, \varepsilon),$$

where $m_0 = -\tau$ and $m_j = \left(\sum_{i=1}^j n_i\right) + (j-1)\tau$ for each $j \geq 1$.

The above definition says that there is some point x whose trajectory shadows each of the (x_i, n_i) in turn, taking a transition time of exactly τ iterates between each one. The numbers m_j for $j \geq 1$ are the time taken for x to shadow (x_1, n_1) up to (x_j, n_j) .

It is sometimes convenient to consider collections \mathcal{G} in which only long orbit segments have specification.

Definition 2.5. *A collection of orbit segments $\mathcal{G} \subset X \times \mathbb{N}$ has tail specification at scale ε if there exists $N_0 \in \mathbb{N}$ so that the collection $\mathcal{G}_{\geq N_0} := \{(x, n) \in \mathcal{G} \mid n \geq N_0\}$ has specification at scale ε .*

2.3.3. *Regularity.* We require the following regularity condition for the potential φ on the collection \mathcal{G} .

Definition 2.6. *Given $\mathcal{G} \subset X \times \mathbb{N}$, a potential φ has the Bowen property on \mathcal{G} at scale ε if*

$$V(\mathcal{G}, \varphi, \varepsilon) := \sup\{|S_n\varphi(x) - S_n\varphi(y)| : (x, n) \in \mathcal{G}, y \in B_n(x, \varepsilon)\} < \infty.$$

We say φ has the Bowen property on \mathcal{G} if there exists $\varepsilon > 0$ so that φ has the Bowen property on \mathcal{G} at scale ε .

If \mathcal{G} has the Bowen property at scale ε , it has it for all smaller scales.

2.4. **General results on uniqueness of equilibrium states.** We prove existence and uniqueness of equilibrium states by using Theorem 5.6 of [19]. The idea is to find a collection of orbit segments $\mathcal{G} \subset X \times \mathbb{N}$ that satisfies specification and the Bowen property, and that is sufficiently large in an appropriate sense. To make this precise, we need the following definition. We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 2.7. *A decomposition for (X, f) consists of three collections $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times \mathbb{N}_0$ and three functions $p, g, s: X \times \mathbb{N} \rightarrow \mathbb{N}_0$ such that for every $(x, n) \in X \times \mathbb{N}$, the values $p = p(x, n)$, $g = g(x, n)$, and $s = s(x, n)$ satisfy $n = p + g + s$, and*

$$(2.4) \quad (x, p) \in \mathcal{P}, \quad (f^p(x), g) \in \mathcal{G}, \quad (f^{p+g}(x), s) \in \mathcal{S}.$$

Given a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ and $M \in \mathbb{N}$, we write \mathcal{G}^M for the set of orbit segments (x, n) for which $p \leq M$ and $s \leq M$.

Note that the symbol $(x, 0)$ denotes the empty set, and the functions p, g, s are permitted to take the value zero.

Theorem 2.8 (Theorem 5.6 of [19]). *Let X be a compact metric space and $f: X \rightarrow X$ a homeomorphism. Let $\varphi: X \rightarrow \mathbb{R}$ be a continuous*

potential function. Suppose there exists $\varepsilon > 0$ such that $P_{\text{exp}}^{\perp}(\varphi, 100\varepsilon) < P(\varphi)$ and $X \times \mathbb{N}$ admits a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ with the following properties:

- (1) For each $M \geq 0$, \mathcal{G}^M has tail specification at scale ε ;
- (2) φ has the Bowen property at scale 100ε on \mathcal{G} ;
- (3) $P(\mathcal{P} \cup \mathcal{S}, \varphi, \varepsilon) + \text{Var}(\varphi, 100\varepsilon) < P(\varphi)$.

Then there is a unique equilibrium state for φ .

We comment on these hypotheses. The transition time τ for specification for \mathcal{G}^M depends on M . If \mathcal{G} had specification at all scales, then a simple argument [19, Lemma 2.10] based on modulus of continuity of f shows that the first hypothesis of the theorem is true for any ε . Thus, considering \mathcal{G}^M for all M at a fixed scale stands in for controlling \mathcal{G} at all scales. The Bonatti–Viana example is a situation where we do not expect to find \mathcal{G} with specification at all scales, but where specification for \mathcal{G}^M for all M at a fixed scale is verifiable.

There are two scales present in the theorem: ε and 100ε . We require specification at scale ε , while expansivity and the Bowen property are controlled at the larger scale 100ε . There is nothing fundamental about the constant 100, but it is essential that expansivity and the Bowen property are controlled at a larger scale than specification. This is because every time we use specification in our argument to estimate an orbit, we move distance up to ε away from our original orbit, and we need to control expansivity and regularity properties for orbits after multiple applications of the specification property. The $\text{Var}(\varphi, 100\varepsilon)$ term appears because we must control points that are distance up to 100ε from a separated set for $\mathcal{P} \cup \mathcal{S}$.

3. PERTURBATIONS OF ANOSOV DIFFEOMORPHISMS

In this section, we collect some more background material about weak forms of hyperbolicity, and perturbations of Anosov diffeomorphisms. We also establish a pressure estimate for C^0 perturbations of Anosov diffeomorphisms that plays a key role in our results.

3.1. Dominated splittings. Let M be a compact manifold and let $f: M \rightarrow M$ be a diffeomorphism. A Df -invariant vector bundle $E \subseteq TM$ has a *dominated splitting* if

$$E = E_1 \oplus \cdots \oplus E_k,$$

where each subbundle E_i is Df -invariant with constant dimension, and there exists an integer $\ell \geq 1$ with the following property: for every

$x \in M$, all $i = 1, \dots, (k - 1)$, and every pair of unit vectors $u \in E_1(x) \oplus \dots \oplus E_i(x)$ and $v \in E_{i+1}(x) \oplus \dots \oplus E_k(x)$, it holds that

$$\frac{|Df_x^\ell(u)|}{|Df_x^\ell(v)|} \leq \frac{1}{2}.$$

See for example [38] or [3, Appendix B, Section 1] for some properties of systems with a dominated splitting.

For us, $k = 2$, and we obtain a dominated splitting $TM = E^{cs} \oplus E^{cu}$, and there exist invariant foliations W^{cs} and W^{cu} tangent to E^{cs} and E^{cu} respectively that we call the *centre-stable and centre-unstable foliations*. For $x \in M$ we let $W^\sigma(x)$ be the leaf of the foliation $\sigma \in \{cs, cu\}$ containing x when this is defined. Given $\eta > 0$, we write $W_\eta^\sigma(x)$ for the set of points in $W^\sigma(x)$ that can be connected to x via a path along $W^\sigma(x)$ with length at most η . Suppose W^1, W^2 are foliations of M . The standard notion of local product structure for W^1, W^2 says that for every $x, y \in M$ that are close enough to each other, the local leaves $W_{\text{loc}}^u(x)$ and $W_{\text{loc}}^s(y)$ intersect in exactly one point. Our definition of local product structure additionally keeps track of the scales involved. We say that W^1, W^2 have *local product structure at scale $\eta > 0$ with constant $\kappa \geq 1$* if for every $x, y \in M$ with $\varepsilon := d(x, y) < \eta$, the leaves $W_{\kappa\varepsilon}^1(x)$ and $W_{\kappa\varepsilon}^2(y)$ intersect in a single point.

3.2. Constants associated to Anosov maps. Let $f: M \rightarrow M$ be a transitive Anosov diffeomorphism. A constant that will be important for us is the constant $C = C(f)$ arising from the Anosov shadowing lemma [24], [33, Theorem 1.2.3].

Lemma 3.1 (Anosov Shadowing Lemma). *Let f be a transitive Anosov diffeomorphism. There exists $C = C(f)$ so that if $2\eta > 0$ is an expansivity constant for f , then every $\frac{\eta}{C}$ -pseudo-orbit $\{x_n\}$ for f can be η -shadowed by an orbit $\{y_n\}$ for f .*

Another constant that will appear in our analysis is $L = L(f, \eta)$ associated with the growth of certain partition sums for f . Recall that f is expansive and has the specification property, and let $h = h_{\text{top}}(f)$ be the topological entropy. For any $\eta > 0$ smaller than the expansivity constant for f , there is a constant $L = L(f, \eta)$ so that

$$(3.1) \quad \Lambda_n^{\text{sep}}(M \times \mathbb{N}, 0, \eta; f) \leq Le^{nh}$$

for every n . See, e.g. [7, Lemma 3]. The constant L can be determined explicitly in terms of the transition time in the specification property.

3.3. Partition sums for C^0 perturbations. Let $f: M \rightarrow M$ be a transitive Anosov diffeomorphism of a compact manifold. Using the Anosov shadowing lemma, we show that there is a C^0 -neighborhood \mathcal{U} of f such that for every $g \in \mathcal{U}$, there is a natural map from g to f given by sending a point x to a point whose f -orbit shadows the g -orbit of x . It is a folklore result that this map is a semi-conjugacy when \mathcal{U} is sufficiently small. For example, this follows from the proof of [13, Proposition 4.1]. This allows us to control partition sums of g at large enough scales from above, and the pressure at all scales from below; the following lemma is proved in §8.

Lemma 3.2. *Let f be a transitive Anosov diffeomorphism. Let $C = C(f)$ be the constant from the Anosov shadowing lemma, and $3\eta > 0$ be an expansivity constant for f . If $g \in \text{Diff}(M)$ is such that $d_{C^0}(f, g) < \eta/C$, then:*

- (i) $P(\varphi; g) \geq P(\varphi; f) - \text{Var}(\varphi, \eta)$;
- (ii) $\Lambda_n^{\text{sep}}(\varphi, 3\eta; g) \leq \Lambda_n^{\text{sep}}(\varphi, \eta; f)e^{n \text{Var}(\varphi, \eta)}$.

It follows from (ii) that

$$(3.2) \quad P(\varphi, 3\eta; g) \leq P(\varphi; f) + \text{Var}(\varphi, \eta).$$

However, it may be that $P(\varphi; g)$ is greater than $P(\varphi, 3\eta; g)$ due to the appearance of entropy at smaller scales for g . Nonetheless, we can obtain an upper bound on $P(\varphi; g)$ which involves the tail entropy; Lemma 2.2 and (3.2) together give the bound

$$(3.3) \quad P(\varphi; g) \leq P(\varphi; f) + h_g^*(3\eta) + 2 \text{Var}(\varphi, 3\eta).$$

The pressure of g , and consequently the tail entropy term, can be arbitrarily large for a C^0 perturbation of f . For example, f can be perturbed continuously in a neighborhood of a fixed point to create a whole disc of fixed points, and then composed with a homeomorphism of this disc that has arbitrarily large entropy.

3.4. Pressure estimates. The examples that we consider are obtained as C^0 -perturbations of Anosov maps, where the perturbation is made inside a small neighborhood of some fixed points. Our strategy is to apply the abstract uniqueness results of Theorem 2.8 when \mathcal{P}, \mathcal{S} are orbit segments spending a large proportion of their time near the fixed points. In this section we give an estimate on the pressure carried by such orbit segments. We fix the following data.

- Let $f: M \rightarrow M$ be a transitive Anosov diffeomorphism of a compact manifold, with topological entropy $h = h_{\text{top}}(f)$.
- Let q be a fixed point for f .

- Let 3η be an expansivity constant for f .
- Let $C = C(f)$ be the constant from the shadowing lemma.
- Let $L = L(f, \eta)$ be a constant so that (3.1) holds.

Now we choose g , \mathcal{C} , and φ :

- Let $g: M \rightarrow M$ be a diffeomorphism with $d_{C^0}(f, g) < \eta/C$.
- Let $\rho < 3\eta$.
- Let $r > 0$ be small, and let $\mathcal{C} = \mathcal{C}(q, r; g) = \{(x, n) \in M \times \mathbb{N} : S_n^g \chi_q(x) < nr\}$, where χ_q is the indicator function of $M \setminus B(q, \rho)$.
- Let φ be any continuous function.

We write $H(r) = -r \log r - (1-r) \log(1-r)$. We have the following entropy and pressure estimates on \mathcal{C} .

Theorem 3.3. *Under the assumptions above, we have the inequality*

$$(3.4) \quad h(\mathcal{C}, 6\eta; g) \leq r(h_{\text{top}}(f) + \log L) + H(2r),$$

and the inequality that for any scale $\delta > 0$,

$$(3.5) \quad P(\mathcal{C}, \varphi, \delta; g) \leq (1-r) \sup_{x \in B(q, \rho)} \varphi(x) + r \sup_{x \in M} \varphi(x) + h(\mathcal{C}, \delta; g),$$

and thus it follows that

$$P(\mathcal{C}, \varphi; g) \leq h_g^*(6\eta) + (1-r) \sup_{x \in B(q, \rho)} \varphi(x) + r(\sup_{x \in M} \varphi(x) + h + \log L) + H(2r).$$

Proof. First we prove the entropy estimate (3.4). For each $(x, n) \in \mathcal{C}$, we partition its orbit into segments entirely in $B(q, \rho)$, and segments entirely outside $B(q, \rho)$. More precisely, given $(x, n) \in \mathcal{C}$, let $((x_i, n_i), (y_i, m_i))_{i=1}^\ell$ be the uniquely determined sequence such that

- $x_0 = x$ and $\sum_{i=1}^\ell (n_i + m_i) = n$;
- $g^{n_i}(x_i) = y_i$ and $g^{m_i}(y_i) = x_{i+1}$;
- $x_i \in B_{n_i}(q, \rho)$ (letting $n_0 = 0$ if $x \notin B(q, \rho)$);
- (y_i, m_i) corresponds to an orbit segment entirely contained in $M \setminus B(q, \rho)$ (letting $m_\ell = 0$ if $g^{n-1}x \in B(q, \rho)$).

Note that $\ell = \ell(x, n)$ satisfies $\ell - 1 \leq \sum_{i=1}^\ell m_i = S_n^g \chi(x) < nr$. For $(x, n) \in \mathcal{C}$, let

$$\underline{t}(x, n) = (\ell, \mathbf{m}, \mathbf{n}) = (\ell, (m_1, \dots, m_\ell), (n_1, \dots, n_\ell))$$

be the time data obtained this way. Given $n \in \mathbb{N}$ and $r > 0$, let

$$\mathcal{J}_n^r = \{(\ell, \mathbf{m}, \mathbf{n}) : 1 \leq \ell \leq nr + 1, \sum (m_i + n_i) = n, \sum m_i < nr\}.$$

Writing $(\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}} = \{(x, n) \in \mathcal{C}_n : \underline{t}(x, n) = (\ell, \mathbf{m}, \mathbf{n})\}$, we have

$$\mathcal{C}_n = \bigcup_{(\ell, \mathbf{m}, \mathbf{n}) \in \mathcal{J}_n^r} (\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}}.$$

Thus we can estimate $\Lambda_n^{\text{sep}}(\mathcal{C}_n, 0, 6\eta)$ in terms of $\Lambda_n^{\text{sep}}((\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}}, 0, 6\eta)$ and $\#\mathcal{J}_n^r$.

For the first of these, let $E_n \subset \mathcal{C}_n$ be $(n, 6\eta)$ -separated, and let F_n be maximally $(n, 3\eta)$ -separated, and thus $(n, 3\eta)$ -spanning, for M . Note that if $z_1, z_2 \in (\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}}$, then $d_{n_i}(g^{s_{i-1}}z_1, g^{s_{i-1}}z_2) < 2\rho < 6\eta$ at times s_i which correspond to the orbits entering $B_{n_i}(q, \rho)$; that is, for $s_0 = 0$ and $s_{i-1} = \sum_{j=1}^{i-1} (n_j + m_j)$. Thus, if $z_1, z_2 \in E_n \cap (\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}}$ with $z_1 \neq z_2$, then there exists i with $d(g^i z_1, g^i z_2) > 6\eta$, and the time i can occur only when the orbit segments are outside $B(q, \rho)$. More precisely, let $r_0 = n_1$, $r_1 = n_1 + m_1 + n_2$, and $r_i = \sum_{j=1}^{i+1} n_j + \sum_{j=1}^i m_j$. There must exist i so that $d_{m_i}(g^{r_{i-1}}z_1, g^{r_{i-1}}z_2) > 6\eta$.

We define a map $\pi: (\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}} \rightarrow F_{m_1} \times \cdots \times F_{m_\ell}$ by choosing $\pi_i(z) \in F_{m_i}$ with the property that $d_{m_i}(g^{r_{i-1}}z, \pi_i(z)) \leq 3\eta$. It follows from the above that if $z_1, z_2 \in E_n \cap (\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}}$ with $z_1 \neq z_2$, there exists i with $d_{m_i}(g^{r_{i-1}}z_1, g^{r_{i-1}}z_2) > 6\eta$, and thus $\pi_i(z_1) \neq \pi_i(z_2)$. Thus, the map π is injective.

Recall that L is the constant such that (3.1) holds and that $h = h_{\text{top}}(f)$. Since $d_{C_0}(f, g) < \eta/C$, using Lemma 3.2, we have

$$(3.6) \quad \Lambda_m^{\text{sep}}(M, 0, 3\eta; g) \leq \Lambda_m^{\text{sep}}(M, 0, \eta; f) \leq L e^{mh}.$$

Thus it follows from injectivity of the map π that

$$\Lambda_n^{\text{sep}}((\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}}, 0, 6\eta) \leq \prod_{i=1}^{\ell} \Lambda_{m_i}^{\text{sep}}(M, 0, 3\eta; g) \leq L^\ell e^{(\sum m_i)h} \leq L^{nr+1} e^{nrh},$$

and thus summing over all choices of $\ell, \mathbf{m}, \mathbf{n}$, we obtain

$$\Lambda_n^{\text{sep}}(\mathcal{C}_n, 0, 6\eta) \leq \sum_{(\ell, \mathbf{m}, \mathbf{n}) \in \mathcal{J}_n^r} \Lambda_n^{\text{sep}}((\mathcal{C}_n)_{\ell, \mathbf{m}, \mathbf{n}}, 0, 6\eta) \leq L^{nr+1} (\#\mathcal{J}_n^r) e^{nrh}.$$

Now we observe that given $1 \leq \ell \leq nr+1$, the choice of \mathbf{m}, \mathbf{n} is uniquely determined by choosing $2\ell-1$ elements of $\{0, 1, \dots, n-1\}$, which are the partial sums of m_i and n_i (the times when the trajectory enters or leaves $B(q, \rho)$, denoted by r_i and s_i above). An elementary computation using Stirling's formula or following [16, Lemma 5.8] shows that the number of such $\ell, \mathbf{m}, \mathbf{n}$ is at most

$$\sum_{k=1}^{2nr+1} \binom{n}{k} \leq (2nr+1)(n+1)e^{nH((2nr+1)/n)+1},$$

and so we have

$$\Lambda_n^{\text{sep}}(\mathcal{C}, 0, 6\eta) \leq L^{nr+1} (2nr+1)(n+1)e^{nrh} e^{nH(2r+\frac{1}{n})}.$$

This gives the bound $h(\mathcal{C}, 6\eta; g) \leq r(h_{\text{top}}(f) + \log L) + H(2r)$, which establishes (3.4). The pressure estimate (3.5) follows from (3.4) by observing that for every $(x, n) \in \mathcal{C}$ we have $g^k x \in \overline{B}(q, \rho)$ for at least $(1-r)n$ values of $k \in \{0, 1, \dots, n-1\}$, and so

$$S_n^g \varphi(x) \leq (1-r)n \sup_{x \in B(q, \rho)} \varphi(x) + rn \sup_{x \in M} \varphi(x);$$

this yields the partition sum estimate

$$\Lambda_n^{\text{sep}}(\mathcal{C}, \varphi, \delta; g) \leq \Lambda_n^{\text{sep}}(\mathcal{C}, 0, \delta; g) \exp(n\{(1-r) \sup_{x \in B(q, \rho)} \varphi(x) + r \sup_{x \in M} \varphi(x)\}),$$

which implies (3.5). The third displayed inequality of Theorem 3.3 is immediate from the inequalities (3.4), (3.5) and Lemma 2.2. \square

3.5. Obstructions to expansivity. Our examples will satisfy the following expansivity property:

- [E]** there exist $\varepsilon > 0$, $r > 0$, and fixed points q, q' such that for $x \in M$, if there exists a sequence $n_k \rightarrow \infty$ with $\frac{1}{n_k} S_{n_k}^g \chi_q(x) \geq r$, and a sequence $m_k \rightarrow \infty$ with $\frac{1}{m_k} S_{m_k}^{g^{-1}} \chi_{q'}(x) \geq r$, then $\Gamma_\varepsilon(x) = \{x\}$.

Let g be as in the previous section, and suppose $\varepsilon > 0$ and $r > 0$ are such that **[E]** holds. Then we have the following pressure estimate.

Theorem 3.4. *Under the above assumptions, we have the pressure estimate $P_{\text{exp}}^\perp(\varphi, \varepsilon) \leq P(\mathcal{C}(q, r) \cup \mathcal{C}(q', r), \varphi)$.*

Proof. Write $\chi = \chi_q, \chi' = \chi_{q'}$, $\mathcal{C} = \mathcal{C}(q, r; g)$, $\mathcal{C}' = \mathcal{C}(q', r; g)$. Consider the sets

$$\begin{aligned} A^+ &= \{x : \text{there exists } K(x) \text{ so } \frac{1}{n} S_n^g \chi(x) < r \text{ for all } n > K(x)\}, \\ A^- &= \{x : \text{there exists } K(x) \text{ so } \frac{1}{n} S_n^{g^{-1}} \chi'(x) < r \text{ for all } n > K(x)\}. \end{aligned}$$

Theorem 3.4 is an application of the following theorem, whose proof is based on the Katok pressure formula [28].

Lemma 3.5. *Let $\mu \in \mathcal{M}_e(g)$. If either $\mu(A^+) > 0$ or $\mu(A^-) > 0$, then $h_\mu(g) + \int \varphi d\mu \leq P(\mathcal{C} \cup \mathcal{C}', \varphi)$.*

Proof. Start with the case where $\mu(A^+) > 0$; we show that $h_\mu(g) + \int \varphi d\mu \leq P(\mathcal{C}, \varphi)$. Given $k \in \mathbb{N}$, let $A_k^+ = \{x \in A^+ : K(x) \leq k\}$, and observe that $\mu(\bigcup_k A_k^+) > 0$, so there is some k such that $\mu(A_k^+) > 0$.

Note that for every $n > k$ and $x \in A_k^+$, we have $(x, n) \in \mathcal{C}$. It follows that for every $\delta > 0$ we have

$$(3.7) \quad \Lambda_n^{\text{sep}}(A_k^+, \varphi, \delta; g) \leq \Lambda_n^{\text{sep}}(\mathcal{C}, \varphi, \delta; g).$$

Fix $\alpha \in (0, \mu(A_k^+))$ and consider the quantity

$$s_n(\varphi, \delta, \mu, \alpha; g) = \inf \left\{ \sum_{x \in E} \exp\{S_n^g \varphi(x)\} : \mu \left(\bigcup_{x \in E} \overline{B}_n(x, \delta) \right) \geq \alpha \right\},$$

where the infimum is taken over finite subsets $E \subset X$. The pressure version of the Katok entropy formula [28] states that

$$h_\mu(g) + \int \varphi d\mu = \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\varphi, \delta, \mu, \alpha; g).$$

Note that $s_n(\varphi, \delta, \mu, \alpha; g) \leq \Lambda_n^{\text{span}}(A_k^+, \varphi, \delta; g) \leq \Lambda_n^{\text{sep}}(A_k^+, \varphi, \delta; g) \leq \Lambda_n^{\text{sep}}(\mathcal{C}, \varphi, \delta; g)$. It follows that

$$h_\mu(g) + \int \varphi d\mu \leq P(\mathcal{C}, \varphi) = \lim_{\delta \rightarrow 0} P(\mathcal{C}, \varphi, \delta).$$

The case where $\mu(A^-) > 0$ is similar: obtain $A_k^- \subset A^-$ such that $K(x) \leq k$ for all $x \in A_k^-$ and $\mu(A_k^-) > 0$. Then observe that for $x \in A_k^-$, we have $(g^{-n}x, n) \in \mathcal{C}'$ for any $n \geq k$. Moreover, (n, ε) -separated sets for g are in one to one correspondence with (n, ε) -separated sets for g^{-1} , and $S_n^{g^{-1}} \varphi(x) = S_n^g \varphi(g^{-n+1}x)$. Then a simple argument shows that $P(A_k^-, \varphi, \varepsilon; g^{-1}) \leq P(\mathcal{C}', \varphi, \varepsilon; g)$.

Finally, Katok's pressure formula applied to g^{-1} tells us that

$$h_\mu(g) + \int \varphi d\mu = \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\varphi, \delta, \mu, \alpha; g^{-1}).$$

Thus $h_\mu(g) + \int \varphi d\mu \leq \lim_{\delta \rightarrow 0} P(A_k^-, \varphi, \varepsilon; g^{-1}) \leq \lim_{\delta \rightarrow 0} P(\mathcal{C}', \varphi, \delta)$. \square

Now, to prove Theorem 3.4, by the hypothesis [E], if $\Gamma_\varepsilon(x) \neq \{x\}$, then either there are only finitely many n so that $\frac{1}{n} S_n^g \chi(x) \geq r$, or there are only finitely many n so that $\frac{1}{n} S_n^{g^{-1}} \chi'(x) \geq r$. Thus, if $x \in \text{NE}(\varepsilon)$, then either $x \in A^+$ or $x \in A^-$. Thus, if μ is an ergodic measure satisfying $\mu(\text{NE}(\varepsilon)) > 0$; then at least one of A^+ or A^- has positive μ -measure. Thus, Theorem 3.5 applies, and we conclude that

$$h_\mu(g) + \int \varphi d\mu \leq P(\mathcal{C} \cup \mathcal{C}', \varphi). \quad \square$$

3.6. Cone estimates and local product structure. Let $F^1, F^2 \subset \mathbb{R}^d$ be subspaces such that $F^1 \cap F^2 = \{0\}$ (we do not assume that $F^1 + F^2 = \mathbb{R}^d$). Let $\angle(F^1, F^2) := \min\{\angle(v, w) : v \in F^1, w \in F^2\}$, and consider the quantity $\bar{\kappa}(F^1, F^2) := (\sin \angle(F^1, F^2))^{-1} \geq 1$. Some elementary trigonometry shows that

$$(3.8) \quad \|v\| \leq \bar{\kappa}(F^1, F^2) \text{ for every } v \in F^1 \text{ with } d(v, F^2) \leq 1,$$

or equivalently,

$$(3.9) \quad \|v\| \leq \bar{\kappa}(F^1, F^2)d(v, F^2) \text{ for every } v \in F^1.$$

Given $\beta \in (0, 1)$ and $F^1, F^2 \subset \mathbb{R}^d$, the β -cone of F^1 and F^2 is

$$C_\beta(F^1, F^2) = \{v + w : v \in F^1, w \in F^2, \|w\| < \beta\|v\|\}.$$

Lemma 3.6. *Let W^1, W^2 be any foliations of $F^1 \oplus F^2$ with C^1 leaves such that $T_x W^1(x) \subset C_\beta(F^1, F^2)$ and $T_x W^2(x) \subset C_\beta(F^2, F^1)$, and let $\bar{\kappa} = \bar{\kappa}(F^1, F^2)$. Then for every $x, y \in F^1 \oplus F^2$ the intersection $W^1(x) \cap W^2(y)$ consists of a single point z . Moreover,*

$$\max\{d_{W^1}(x, z), d_{W^2}(y, z)\} \leq \frac{1 + \beta}{1 - \beta} \bar{\kappa} d(x, y).$$

We will consider foliations on \mathbb{T}^4 whose lifts to \mathbb{R}^4 satisfy the hypotheses of Lemma 3.6. Uniqueness of the intersection point on \mathbb{T}^4 follows from restricting to sufficiently small local leaves. We also need the following lemma, which compares the intrinsic distance along a leaf with the distance induced from the metric on \mathbb{T}^4 .

Lemma 3.7. *Under the assumptions of Lemma 3.6, suppose that x, y are points belonging to the same local leaf of $W \in \{W^1, W^2\}$. Then*

$$d(x, y) \leq d_W(x, y) \leq (1 + \beta)^2 d(x, y).$$

Lemmas 3.6 and 3.7 are proved in §8.

4. BONATTI-VIANA CONSTRUCTION AND MAIN RESULT

In [4], Bonatti and Viana defined a C^1 -open class of diffeomorphisms by a list of 4 hypotheses, which ensure robust transitivity and the existence of a dominated splitting into two bundles with no invariant sub-bundles. They then gave an explicit construction of a family of diffeomorphisms satisfying these 4 hypotheses, thus showing that the Bonatti-Viana class is non-empty. We refer to this as the *Bonatti-Viana construction*. The diffeomorphisms constructed this way, and their C^1 perturbations are the object of our study. We recall the main points of the construction, referring to [4] and [12] for full details. In [12], Buzzi and Fisher added some refinements to the details of the construction, allowing useful additional control which we shall assume in this paper. Let $A \in \text{SL}(4, \mathbb{Z})$ with four distinct real eigenvalues

$$0 < \lambda_1 < \lambda_2 < 1/3 < 3 < \lambda_3 < \lambda_4.$$

The Bonatti–Viana construction yields diffeomorphisms, which we denote by f_{BV} , which are C^0 small, but C^1 large, deformations of f_A .

Recall that 3η is an expansivity constant for f_A . At some points in our analysis (see §6.1 and §6.4), we require that η is not too large so that calculations at scales involving η are local. We fix $0 < \rho < 3\eta$ and carry out a perturbation in ρ -neighbourhoods of q and q' . Around q we will deform in the weak stable direction and around q' in the weak unstable direction. The third fixed point will be left unperturbed to ensure robust transitivity.

Let F^s, F^u be the two-dimensional subspaces of \mathbb{R}^d corresponding to contracting and expanding eigenvalues of A , respectively. Let $\kappa = 2\bar{\kappa}(F^s, F^u)$, where $\bar{\kappa}$ is as in (3.8).

Fixing $\rho > 0$, we consider the scales $\rho' = 5\rho$ and $\rho'' = 300\kappa\rho'$. We assume that ρ is sufficiently small that $\rho'' < 6\eta$. The role of these scales is as follows:

- (1) The perturbation takes place in the balls $B(q, \rho)$ and $B(q', \rho)$ – outside of these balls the new map is identically equal to f_A ;
- (2) The scale ρ' is chosen so at this scale the center-stable (resp. center-unstable) leaves are contracted by g (resp. g^{-1});
- (3) The scale ρ'' is the distance that points need to be away from q and q' to guarantee uniform contraction/expansion estimates at a large enough scale to verify the hypotheses of Theorem 2.8.

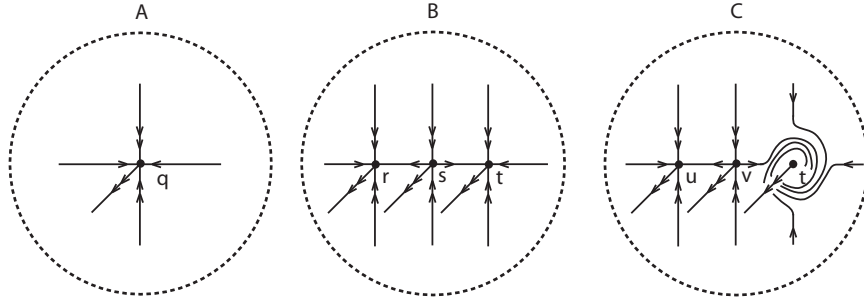


FIGURE 1. Bonatti–Viana construction

The deformation around the points q and q' is done in two steps, illustrated in Figure 1. We describe the deformation around q . First, we perform a deformation around q in the stable direction λ_2 as follows. Inside $B(q, \rho)$, the fixed point q undergoes a pitchfork bifurcation in the direction corresponding to λ_2 .

The stable index of q changes from 2 to 1 and two new fixed points q_1 and q_2 are created. The second step is to deform the diffeomorphism in a neighborhood of q_2 so that the contracting eigenvalues become complex; see Figure 1.

Note the creation of fixed points with different indices prevents the topologically transitive map from being Anosov. These non-real eigenvalues also forbid the existence of a one-dimensional invariant subbundle inside E^{cs} . So the resulting map \hat{f} has a splitting $E^{cs} \oplus E^{cu}$.

To finish the construction take the deformation just made on f_A near q and repeat it so that the map is equal to \hat{f}^{-1} in the neighborhood of q' . We obtain a map f_{BV} that is robustly transitive, not partially hyperbolic, and has a dominated splitting $T\mathbb{T}^4 = E^{cs} \oplus E^{cu}$ with $\dim E^{cs} = \dim E^{cu} = 2$ (see [4] for proofs of these facts).

We fix a small β and we can ensure in the construction that $E^{cs} \subset C_{\beta/2}(F^s, F^u)$ and $E^{cu} \subset C_{\beta/2}(F^u, F^s)$. To simplify computations, we assume explicit upper bounds on β at a couple of points in the proof (see e.g. proof of Lemma 6.3). We may assume that $\beta < 1/3$.

Let $C = C(f_A)$ be the constant provided by Lemma 3.1. Outside $B(q, \rho) \cup B(q', \rho)$, the maps f_{BV} and f_A are identical, and we can carry out the construction so there exists a constant K so that both $f_A(B(q, \rho)) \subset B(q, K\rho)$ and $f_{BV}(B(q, \rho)) \subset B(q, K\rho)$, and similarly for q' . Thus the C^0 distance between f_{BV} and f_A is at most $K\rho$. In particular, by choosing ρ small, we can ensure that $d_{C^0}(f_{BV}, f_A) < \eta/C$. This allows us to apply Lemma 3.2 to f_{BV} , or to a perturbation of f_{BV} .

We now consider diffeomorphisms g in a C^1 neighborhood of f_{BV} . We recall results from [12] on integrability of foliations. We assume that the construction of f_{BV} is carried out so that the resulting deformation *respects the domination of f_A* . This property is defined in [12, Definition 2.3], and verified for f_{BV} in [12, §7]. This is a C^1 robust condition which, by Theorem 3.1 of [12], ensures integrability of the dominated splitting. Thus, for $g \in \text{Diff}(\mathbb{T}^4)$ sufficiently close to f_{BV} , there are invariant foliations tangent to E_g^{cs} and E_g^{cu} respectively. Furthermore, the argument of Lemma 6.1 and 6.2 of [4] shows that each leaf of each foliation is dense in the torus. The existence of foliations was not known when [4] was written, but these arguments apply with only minor modification now that the existence result has been established by [12]. Thus, we can consider a C^1 -neighborhood \mathcal{V} of f_{BV} such that the following is true for every $g \in \mathcal{V}(f_{BV})$:

- $d_{C^0}(g, f_A) < \eta/C$;
- g has a dominated splitting $T\mathbb{T}^4 = E_g^{cs} \oplus E_g^{cu}$, with $\dim E_g^{cs} = \dim E_g^{cu} = 2$ and E_g^{cs}, E_g^{cu} contained in $C_\beta(F^s, F^u)$ and $C_\beta(F^u, F^s)$ respectively;
- The distributions E_g^{cs}, E_g^{cu} integrate to foliations W_g^{cs}, W_g^{cu} .
- Each of the leaves $W_g^{cs}(x)$ and $W_g^{cu}(x)$ is dense for every $x \in \mathbb{T}^4$.

Given $g \in \mathcal{V}$, we define the quantities

$$\begin{aligned}\lambda_s(g) &= \sup\{\|Dg|_{E^{cs}(x)}\| : x \in \mathbb{T}^4 \setminus B(q, \rho)\}, \\ \lambda_u(g) &= \inf\{\|Dg|_{E^{cu}(x)}^{-1}\|^{-1} : x \in \mathbb{T}^4 \setminus B(q', \rho)\}, \\ \lambda_{cs}(g) &= \sup\{\|Dg|_{E^{cs}(x)}\| : x \in \mathbb{T}^4\}, \\ \lambda_{cu}(g) &= \inf\{\|Dg|_{E^{cu}(x)}^{-1}\|^{-1} : x \in \mathbb{T}^4\}, \\ \lambda(g) &= \max\{\lambda_{cs}(g), \lambda_{cu}(g)^{-1}\}.\end{aligned}$$

Note that by the construction of f_{BV} we have

$$\begin{aligned}\lambda_s(f_{BV}) &< 1 < \lambda_{cs}(f_{BV}), \\ \lambda_{cu}(f_{BV}) &< 1 < \lambda_u(f_{BV}),\end{aligned}$$

and we can carry out the construction so that $\lambda(f_{BV})$ is arbitrarily close to 1. By continuity, these inequalities hold for C^1 -perturbations of f_{BV} . We also have $\lambda_s(g)$ and $\lambda_u(g)$ arbitrarily close to the corresponding values for f_A . We let

$$(4.1) \quad \gamma(g) = \max\left\{\frac{\log \lambda_{cs}(g)}{\log \lambda_{cs}(g) - \log \lambda_s(g)}, \frac{\log \lambda_{cu}(g)}{\log \lambda_{cu}(g) - \log \lambda_u(g)}\right\}$$

Note that $\gamma(g) \rightarrow 0$ as $\lambda(g) \rightarrow 1$ (as long as $\lambda_s(g), \lambda_u(g) \not\rightarrow 1$). A simple calculation shows that for any $r > \gamma$, we have

$$(4.2) \quad \lambda_{cs}^{1-r} \lambda_s^r < 1,$$

$$(4.3) \quad \lambda_{cu}^{1-r} \lambda_u^r > 1,$$

so that in particular, writing

$$(4.4) \quad \theta_r(g) = \min(\lambda_{cs}^{1-r} \lambda_s^r, \lambda_{cu}^{-(1-r)} \lambda_u^{-r}),$$

we have $\theta_r(g) < 1$ for all $r > \gamma(g)$. For notational convenience, we write

$$(4.5) \quad Q = B(q, \rho'' + \rho) \cup B(q', \rho'' + \rho).$$

We now state the precise version of Theorem A.

Theorem 4.1. *Given $g \in \mathcal{V}(f_{BV})$ as above, let $\gamma = \gamma(g)$, $\lambda = \lambda(g)$. Let $\varphi: \mathbb{T}^4 \rightarrow \mathbb{R}$ be Hölder continuous, and set $V = \text{Var}(\varphi, 300\rho')$. Let*

$$(4.6) \quad \Phi(\varphi; g) = 6 \log \lambda + (1 - \gamma) \sup_Q \varphi + \gamma (\sup_{\mathbb{T}^4} \varphi + \log L + h) + H(2\gamma) + V.$$

If $\Phi(\varphi; g) < P(\varphi; g)$, then φ has a unique equilibrium state.

The C^1 -open set $\bigcup_{f_{BV} \in \mathcal{F}_{\lambda, \rho}} \mathcal{V}(f_{BV})$ gives a large class of Bonatti-Viana diffeomorphisms for which this theorem can be used to investigate uniqueness of equilibrium states.

5. COROLLARIES OF THEOREM 4.1

Before we prove Theorem 4.1, we show how to use it to obtain the two corollaries mentioned in the introduction.

Corollary 5.1. *Let $\mathcal{V}(f_{BV}) \subset \text{Diff}(\mathbb{T}^4)$ be as above, and suppose $g \in \mathcal{V}(f_{BV})$ is such that for $L = L(f_A, \eta)$, $h = h_{\text{top}}(f_A)$, $\gamma = \gamma(g)$, and $\lambda = \lambda(g)$ we have*

$$(5.1) \quad 6 \log \lambda + \gamma(\log L + h) + H(2\gamma) < h.$$

Let $V(\varphi) = \text{Var}(\varphi, 300\rho') + \text{Var}(\varphi, \eta')$, where $\eta' = C(f_A)d_{C^0}(f_A, g)$. Then writing

$$D = h - 6 \log \lambda - \gamma(\log L + h) - H(2\gamma) > 0,$$

every Hölder continuous potential φ with the bounded range hypothesis $\sup \varphi - \inf \varphi + V(\varphi) < D$ has a unique equilibrium state. In particular, (5.1) is a sufficient criterion for $g \in \mathcal{V}(f_{BV})$ to have a unique MME.

Proof. If $\sup \varphi - \inf \varphi + V(\varphi) < D$, then

$$\begin{aligned} & 6 \log \lambda + (1 - \gamma) \sup_Q \varphi + \gamma(\sup_{\mathbb{T}^4} \varphi + h + \log L) + H(2\gamma) + V \\ &= (1 - \gamma) \sup_Q \varphi + \gamma(\sup_{\mathbb{T}^4} \varphi) + h_{\text{top}}(f_A) + V - D \\ &\leq \sup_{\mathbb{T}^4} \varphi + h_{\text{top}}(f_A) + V - D \\ &< \inf \varphi + h_{\text{top}}(f_A) - \text{Var}(\varphi, \eta') \\ &\leq P(\varphi; f_A) - \text{Var}(\varphi, \eta') \leq P(\varphi; g). \end{aligned}$$

Thus Theorem 4.1 applies. \square

Since $V(\varphi) \leq 2(\sup \varphi - \inf \varphi)$, we could remove the variance term in our bounded range hypothesis by asking that $3(\sup \varphi - \inf \varphi) < D$.

Corollary 5.2. *Let $\varphi: \mathbb{T}^4 \rightarrow \mathbb{R}$ be a Hölder continuous potential. In any C^0 -neighborhood of f_A , there exists a C^1 -open subset $\mathcal{V}_0 \subset \text{Diff}(\mathbb{T}^4)$ containing diffeomorphisms from the Bonatti–Viana family such that for every $g \in \mathcal{V}_0$, g has a dominated splitting and is not partially hyperbolic and $(\mathbb{T}^4, g, \varphi)$ has a unique equilibrium state.*

Proof. A diffeomorphism $f_{BV} \in \mathcal{F}_{\lambda, \rho}$ can be found in any C^0 neighbourhood of f_A by taking ρ to be small. Let $\mathcal{V} = \mathcal{V}(f_{BV})$, and \mathcal{V}_0 be the set of $g \in \mathcal{V}$ such that $\Phi(g; \varphi) < P(\varphi; g)$. Note that $\Phi(g; \varphi)$ is continuous under C^1 perturbation of g , so \mathcal{V}_0 is C^1 -open. It only remains to show that \mathcal{V}_0 is non-empty when ρ and $\log \lambda$ are sufficiently small.

Let $\eta' = C(f_A)d_{C_0}(g, f_A)$. Recall from Lemma 3.2(i) that $P(\varphi; g) \geq P(\varphi; f_A) - \text{Var}(\varphi, \eta')$. Moreover, we have

$$(1 - \gamma) \sup_Q \varphi \leq \max\{\varphi(q), \varphi(q')\} + \text{Var}(\varphi, 2\rho'').$$

Thus to prove $\Phi(\varphi; g) < P(\varphi; g)$ it suffices to verify that

$$\max\{\varphi(q), \varphi(q')\} + 6 \log \lambda + \gamma(\sup_{\mathbb{T}^4} \varphi + h + \log L) + H(\gamma) + V' < P(\varphi; f_A),$$

where $V' = V + \text{Var}(\varphi, 2\rho'') + \text{Var}(\varphi, \eta')$. The scales which appear in the V' term all tend to 0 as ρ tends to 0. Given a hyperbolic toral automorphism f_A and a Hölder potential $\varphi: \mathbb{T}^4 \rightarrow \mathbb{R}$, it is well known that φ has a unique equilibrium state with the Gibbs property. For a fixed point p , the Dirac measure δ_p clearly does not have the Gibbs property, so cannot be an equilibrium state for φ , and thus

$$\varphi(p) = h_{\delta_p}(f_A) + \int \varphi d\delta_p < P(\varphi; f_A).$$

Thus, $\max\{\varphi(q), \varphi(q')\} < P(\varphi; f_A)$. By choosing $\log \lambda$ and ρ small, we can ensure that γ and V' are small enough so that the required inequality holds. Thus, \mathcal{V}_0 is non-empty. \square

6. PROOF OF THE MAIN RESULT

We now build up a proof of our main result Theorem 4.1, which is the more precise version of Theorem A.

6.1. Local product structure. We now establish local product structure at scale 6η for maps $g \in \mathcal{V}$. The assumptions that allow us to do this are that $E_g^\sigma \subset C_\beta^\sigma$ for $\sigma \in \{cu, cs\}$ and that β, η are not too large.

Lemma 6.1. *Every $g \in \mathcal{V}$ has a local product structure for W_g^{cs}, W_g^{cu} at scale 6η with constant $\kappa = 2\bar{\kappa}(F^s, F^u)$.*

Proof. Let \widetilde{W}^{cs} and \widetilde{W}^{cu} be the lifts of W_g^{cs}, W_g^{cu} to \mathbb{R}^4 . Given $x, y \in \mathbb{T}^4$ with $\varepsilon := d(x, y) < 6\eta$, let $\tilde{x}, \tilde{y} \in \mathbb{R}^4$ be lifts of x, y with $\varepsilon = d(\tilde{x}, \tilde{y}) < 6\eta$. By Lemma 3.6 the intersection $\widetilde{W}^{cs}(x) \cap \widetilde{W}^{cu}(y)$ has a unique point \tilde{z} , which projects to $z \in \mathbb{T}^4$. Moreover, the leaf distances between \tilde{x}, \tilde{z} and \tilde{y}, \tilde{z} are at most $(\frac{1+\beta}{1-\beta})\bar{\kappa}(F^s, F^u)\varepsilon$. Since $\beta < \frac{1}{3}$ this is less than $2\bar{\kappa}(F^s, F^u)\varepsilon$, so z is in the intersection of the local leaves $(W_g^{cs})_{\kappa\varepsilon}(x)$ and $(W_g^u)_{\kappa\varepsilon}(x)$.

By choosing η not too large, we can ensure that $6\eta\kappa$ is not too large relative to the diameter of \mathbb{T}^4 , so that the projection of $\widetilde{W}_{6\eta\kappa}^{cs}(x) \cap \widetilde{W}_{6\eta\kappa}^{cu}(y)$ coincides with $W_{6\eta\kappa}^{cs}(x) \cap W_{6\eta\kappa}^{cu}(y)$. Thus, z is the only point in this intersection. \square

6.2. Specification. We must control the size of local leaves of W^{cs} , W^{cu} under iteration, and the time to transition from one orbit to another. We use the following fact, which we prove in §8.

Lemma 6.2. *For every $\delta > 0$ there is $R > 0$ such that for all $x, y \in \mathbb{T}^4$, we have $W_R^{cu}(x) \cap W_\delta^{cs}(y) \neq \emptyset$.*

Although the leaves $W^{cu}(x)$ are not expanding at every point, and the leaves $W^{cs}(x)$ are not contracting at every point, we nevertheless see expansion and contraction if we look at a scale suitably large relative to ρ . More precisely, consider the quantities $\theta_{cs} = \frac{4}{5} + \frac{1}{5}\lambda_s(g) < 1$ and $\theta_{cu} = \frac{4}{5} + \frac{1}{5}\lambda_u(g)^{-1} < 1$. Let d_{cs} and d_{cu} be the metrics on the leaves W^{cs} and W^{cu} . Then we have the following result.

Lemma 6.3. *If $x \in \mathbb{T}^4$ and $y \in W^{cs}(x)$ are such that $d_{cs}(x, y) > \rho'$, then $d_{cs}(gx, gy) < \theta_{cs}d_{cs}(x, y)$. Similarly, if $y \in W^{cu}(x)$ and $d_{cu}(x, y) > \rho'$, then $d_{cu}(g^{-1}x, g^{-1}y) < \theta_{cu}d_{cu}(x, y)$.*

Proof. We give the proof for W^{cs} ; the proof for W^{cu} is analogous. Given a path σ on \mathbb{T}^4 , write $\ell(\sigma)$ for the length of σ . Let σ be a path from x to y in $W^{cs}(x)$ such that $\ell(\sigma) = d_{cs}(x, y)$. Decompose σ as the disjoint union of paths σ_i where $\ell(\sigma_i) \in [\rho', 2\rho']$. Clearly it suffices to show that $\ell(g\sigma_i) < \theta_{cs}\ell(\sigma_i)$ for each i . We may assume that β is chosen not too large so that

$$(6.1) \quad (1 + \beta) \left(\frac{\lambda(g) - \lambda_s(g)}{1 - \lambda_s(g)} \right) < 2$$

We may assume that the path σ_i has at most one connected component that intersects $B(q, \rho)$, since ρ and $\ell(\sigma_i) \leq 2\rho'$ are not large enough to wrap around the torus. Let ℓ_1 be the length of this component; because this component lies in $W^{cs}(x)$, which is contained in $C_\beta(F^s, F^u)$, we have $\ell_1 \leq 2\rho(1 + \beta)$. Let $\ell_2 = \ell(\sigma_i) - \ell_1$. Let v be a tangent vector to the curve σ at the point $p \in \mathbb{T}^4$. If $p \in B(q, \rho)$ then we have $\|Dg(v)\| \leq \lambda(g)\|v\|$, while if $p \notin B(q, \rho)$ then $\|Dg(v)\| \leq \lambda_s(g)\|v\|$. Thus we obtain

$$\begin{aligned} \ell(g\sigma_i) &\leq \lambda\ell_1 + \lambda_s\ell_2 = (\lambda - \lambda_s)\ell_1 + \lambda_s\ell(\sigma_i) \\ &\leq (\lambda - \lambda_s)2\rho(1 + \beta) + \lambda_s\ell(\sigma_i) < 4(1 - \lambda_s)\rho + \lambda_s\ell(\sigma_i), \end{aligned}$$

where the last inequality uses (6.1). Since $\rho = \frac{1}{5}\rho' \leq \frac{1}{5}\ell(\sigma_i)$, this gives

$$\ell(g\sigma_i) < \frac{4}{5}(1 - \lambda_s)\ell(\sigma_i) + \lambda_s\ell(\sigma_i) = \theta_{cs}\ell(\sigma_i).$$

Summing over i gives $d_{cs}(gx, gy) \leq \ell(g\sigma) < \theta_{cs}\ell(\sigma) = \theta_{cs}d_{cs}(x, y)$. The proof for d_{cu} is similar. \square

The following is an immediate consequence of Lemmas 6.3 and 6.2.

Lemma 6.4. *For every $R > \rho'$ and $x \in \mathbb{T}^4$, we have*

$$\begin{aligned} g(W_R^{cs}(x)) &\subset W_{\theta_{cs}R}^{cs}(gx), \\ g^{-1}(W_R^{cu}(x)) &\subset W_{\theta_{cu}^{-1}R}^{cu}(g^{-1}x). \end{aligned}$$

In particular, there is $\tau_0 \in \mathbb{N}$ such that for every $x, y \in \mathbb{T}^4$ we have

$$(6.2) \quad g^{\tau_0}(W_{\rho'}^{cu}(x)) \cap W_{\rho'}^{cs}(y) \neq \emptyset.$$

Let $\rho'' := 300\kappa\rho'$, where $\kappa = 2\bar{\kappa}(F^s, F^u)$ is the constant arising in the local product structure of W^{cs}, W^{cu} . Let χ be the indicator function of $\mathbb{T}^4 \setminus B(q, \rho'' + \rho)$ and χ' be the indicator function of $\mathbb{T}^4 \setminus B(q', \rho'' + \rho)$. The scale $\rho'' + \rho$ is chosen to ensure uniform estimates on $W_{\rho''}^{cs}$ and $W_{\rho''}^{cu}$ for points with $\chi(x) = 1$ and $\chi'(x) = 1$.

From now on we fix $r > \gamma(g)$, and consider the following collection of orbit segments:

$$\mathcal{G} = \{(x, n) : \frac{1}{i}S_i\chi(x) \geq r \text{ and } \frac{1}{i}S_i\chi'(f^{n-i}x) \geq r \text{ for all } 0 \leq i \leq n\}.$$

We will show that \mathcal{G}^M has specification at scale $3\rho'$. To get a decomposition we consider \mathcal{G} together with the collections

$$\begin{aligned} \mathcal{P} &= \{(x, n) \in \mathbb{T}^4 \times \mathbb{N} : \frac{1}{n}S_n\chi(x) < r\}, \\ \mathcal{S} &= \{(x, n) \in \mathbb{T}^4 \times \mathbb{N} : \frac{1}{n}S_n\chi'(x) < r\}. \end{aligned}$$

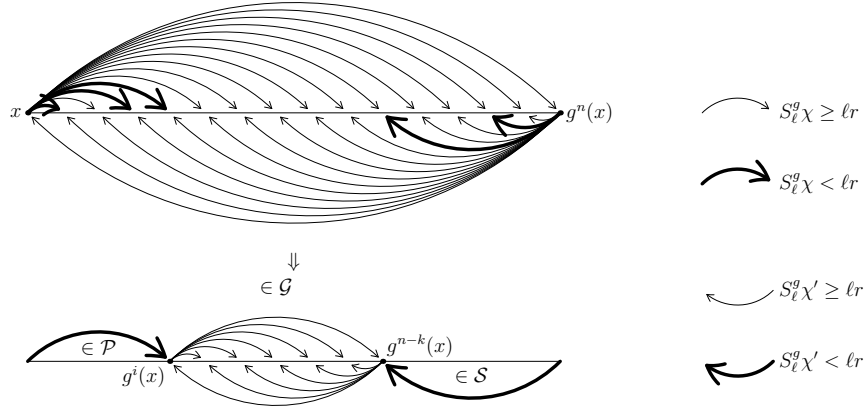


FIGURE 2. Decomposing an orbit segment

Lemma 6.5. *The collections $\mathcal{P}, \mathcal{G}, \mathcal{S}$ form a decomposition for g .*

Proof. Let $(x, n) \in X \times \mathbb{N}$. Let $0 \leq i \leq n$ be the largest integer so $\frac{1}{i}S_i\chi(x) < r$, and $0 \leq k \leq n$ be the largest integer so $\frac{1}{k}S_k\chi'(g^{n-k}x) < r$. A short calculation shows that $\frac{1}{\ell}S_\ell\chi(g^i x) \geq r$ for $0 \leq \ell \leq n - i$, and

$\frac{1}{\ell} S_\ell \chi'(g^{n-k-\ell}x) \geq r$ for $0 \leq \ell \leq n-k$, see Figure 2. Thus, if we assume that $i+k < n$, letting $j = n - (i+k)$, we have

$$(x, i) \in \mathcal{P}, \quad (g^i x, j) \in \mathcal{G}, \quad (g^{i+j} x, k) \in \mathcal{S}.$$

If $i+k \geq n$, we can choose a decomposition with $j = 0$. \square

Orbit segments in \mathcal{G}^M , which is the set of orbit segments (x, n) for which $p \leq M$ and $s \leq M$, satisfy the following.

Lemma 6.6. *Let $\nu = \lambda/\theta_r$. For every $M \in \mathbb{N}_0$, $(x, n) \in \mathcal{G}^M$, and $0 \leq i \leq n$, we have*

- (a) $\|Dg^i|_{E^{cs}(y)}\| \leq \nu^{2M} \theta_r^i$ for $y \in B_n(x, \rho'')$;
- (b) $\|Dg^{-i}|_{E^{cu}(g^i y)}\| \leq \nu^{2M} \theta_r^i$ for $y \in B_n(x, \rho'')$;
- (c) $d_{cs}(g^i y, g^i z) \leq \nu^{2M} \theta_r^i d_{cs}(y, z)$ when $y \in B_n(x, \rho')$ and $z \in W_{2\rho'}^{cs}(y)$;
- (d) $d_{cu}(g^{n-i} y, g^{n-i} z) \leq \nu^{2M} \theta_r^i d_{cu}(y, z)$ when $y \in B_n(g^{-n} x, \rho')$ and $z \in W_{2\rho'}^{cu}(y)$.

Proof. We prove (a). Given $(x, n) \in \mathcal{G}^M$ and $0 \leq i \leq n$, we have $S_i \chi(x) > ir - 2M$, and so the orbit segment (x, i) spends greater than $ir - 2M$ iterates outside $B(q, 4\rho')$, and thus (y, i) spends greater than $ir - 2M$ iterates outside $B(q, \rho)$. It follows that

$$\begin{aligned} \|Dg^i|_{E^{cs}(y)}\| &\leq \lambda^{i-(ir-2M)} \lambda_s^{ir-2M} \\ &= \lambda^{i(1-r)} \lambda_s^{ir} \lambda^{2M} \lambda_s^{-2M} = (\theta_r)^i \nu^{2M}. \end{aligned}$$

For (c), note that if $y \in B_n(x, \rho')$ and $z' \in W_{2\rho'}^{cs}(y)$, then $z' \in B_n(x, 3\rho')$. Thus, the uniform derivative estimate of (a) applies to all points in $W_{2\rho'}^{cs}(y)$, and it is an easy exercise to use this to obtain the statement of (c). The proof of (b) is similar to (a), and (d) follows. \square

We use the following facts for our result on the specification property:

- For any $x \in \mathbb{T}^4$ and $n \in \mathbb{N}$, from Lemma 6.4 we have $W_{\rho'}^{cs}(x) \subset B_n(x, \rho')$ and $g^{-n}(W_{\rho'}^{cu}(g^n x)) \subset B_n(x, \rho')$;
- If $(x, n) \in \mathcal{G}^M$ and $y, z \in B_n(x, 3\rho')$ and $g^n z \in W^{cu}(g^n y)$, then Lemma 6.6 (c) gives $d_n(y, z) \leq \nu^{2M} d_{cu}(g^n y, g^n z)$ and $d_{cu}(y, z) \leq \nu^{2M} \theta_r^n d_{cu}(g^n y, g^n z)$.

Given M , we take $N = N(M)$ such that $\theta_r^N \nu^{2M} \lambda^{\tau_0} < \frac{1}{2}$, where τ_0 is as in (6.2). Then we let $\mathcal{G}_{\geq N}^M := \{(x, n) \in \mathcal{G}^M \mid n \geq N\}$.

Lemma 6.7. *For every M , let $N = N(M)$ be as above. Then $\mathcal{G}_{\geq N}^M$ has specification at scale $3\rho'$.*

Proof. Write $\tau = \tau_0$, so that (6.2) gives $g^\tau(W_{\rho'}^{cu}(x)) \cap W_{\rho'}^{cs}(y) \neq \emptyset$ for every $x, y \in \mathbb{T}^4$.

For every $(x, n) \in \mathcal{G}_{\geq N}^M$ and $y, z \in g^{-(n+\tau)}(g^\tau(W_{\rho'}^{cu}(x)))$, our choice of N gives

$$(6.3) \quad d(y, z) < \frac{1}{2}d(g^{n+\tau}y, g^{n+\tau}z).$$

Now we show that $\mathcal{G}_{\geq N}^M$ has specification with transition time τ . Given any $(x_1, n_1), \dots, (x_k, n_k) \in \mathcal{G}^M$ with $n_i \geq N$, we construct y_j iteratively such that (y_j, m_j) shadows $(x_1, n_1), \dots, (x_j, n_j)$, where $m_1 = n_1$, $m_2 = n_1 + \tau + n_2$, \dots , $m_k = (\sum_{i=1}^k n_i) + (k-1)\tau$. We also set $m_0 = -\tau$.

Start by letting $y_1 = x_1$, and we choose y_2, \dots, y_k iteratively so that

$$\begin{array}{ccc} g^{m_1}y_2 \in W_{\rho'}^{cu}(g^{m_1}y_1) & \text{and} & g^{m_1+\tau}y_2 \in W_{\rho'}^{cs}(x_2) \\ g^{m_2}y_3 \in W_{\rho'}^{cu}(g^{m_2}y_2) & \text{and} & g^{m_2+\tau}y_3 \in W_{\rho'}^{cs}(x_3) \\ \vdots & & \vdots \\ g^{m_{k-1}}y_k \in W_{\rho'}^{cu}(g^{m_{k-1}}y_{k-1}) & \text{and} & g^{m_{k-1}+\tau}y_k \in W_{\rho'}^{cs}(x_k). \end{array}$$

That is, for $j \in \{1, \dots, k-1\}$, we let y_{j+1} be a point such that

$$y_{j+1} \in g^{-m_j}(W_{\rho'}^{cu}(g^{m_j}y_j)) \cap g^{-(m_j+\tau)}(W_{\rho'}^{cs}(x_{j+1})).$$

Using the fact that $g^{m_j}y_{j+1}$ is in the centre-unstable manifold of $g^{m_j}y_j$ together with the estimate (6.3), we obtain that

$$\begin{array}{ccc} d_{n_j}(g^{m_{j-1}+\tau}y_j, g^{m_{j-1}+\tau}y_{j+1}) & < & \rho' \\ d_{n_{j-1}}(g^{m_{j-2}+\tau}y_j, g^{m_{j-2}+\tau}y_{j+1}) & < & \rho'/2 \\ \vdots & & \vdots \\ d_{n_1}(y_j, y_{j+1}) & < & \rho'/2^{j-1}. \end{array}$$

That is, $d_{n_{j-i}}(g^{m_{j-i-1}+\tau}y_j, g^{m_{j-i-1}+\tau}y_{j+1}) < \rho'/2^i$ for $i \in \{0, \dots, j-1\}$. This estimate, together with the fact that $g^{m_j+\tau}(y_{j+1}) \in B_{n_{j+1}}(x_{j+1}, \rho')$ from Lemma 6.4 gives that $d_{n_j}(g^{m_{j-1}+\tau}y_k, x_j) < 2\rho' + \sum_{j=1}^{\infty} 2^{-j}\rho' = 3\rho'$. It follows that

$$y_k \in \bigcap_{j=1}^k g^{-(m_{j-1}+\tau)}B_{n_j}(x_j, 3\rho'),$$

and thus $\mathcal{G}_{\geq N}^M$ has specification at scale $3\rho'$. \square

6.3. Bowen property. Let $\theta_r \in (0, 1)$ be the constant that was defined at (4.4), and let κ be the constant associated with the local product structure of $E_g^{cs} \oplus E_g^{cu}$.

Lemma 6.8. *Given $(x, n) \in \mathcal{G}$ and $y \in B_n(x, 300\rho')$, we have*

$$(6.4) \quad d(g^k x, g^k y) \leq \kappa 600\rho'(\theta_r^{n-k} + \theta_r^k)$$

for every $0 \leq k \leq n$.

Proof. Using the local product structure at scale $300\rho'$, there exists $z \in W_{\kappa 300\rho'}^{cs}(x) \cap W_{\kappa 300\rho'}^{cu}(y) = W_{\rho'}^{cs}(x) \cap W_{\rho'}^{cu}(y)$. By Lemma 6.6, we get

$$d(g^k z, g^k y) \leq \theta_r^{n-k} d(g^n z, g^n y) \leq \theta_r^{n-k} \kappa 300\rho',$$

and $d(g^k x, g^k z) \leq \theta_r^k d(x, z) \leq \theta_r^k \kappa 300\rho'$. The result follows. \square

Lemma 6.9. *Any Hölder continuous φ has the Bowen property on \mathcal{G} at scale $300\rho'$.*

Proof. By Hölder continuity, there are constants $K > 0$ and $\alpha \in (0, 1)$ such that $|\varphi(x) - \varphi(y)| \leq K d(x, y)^\alpha$ for all $x, y \in \mathbb{T}^d$. Now given $(x, n) \in \mathcal{G}$ and $y \in B_n(x, 300\rho')$, Lemma 6.8 gives

$$\begin{aligned} |S_n \varphi(x) - S_n \varphi(y)| &\leq K \sum_{k=0}^{n-1} d(g^k x, g^k y)^\alpha \leq K \kappa 600\rho' \sum_{k=0}^{n-1} (\theta_r^{n-k} + \theta_r^k)^\alpha \\ &\leq 2^\alpha K \kappa 600\rho' \sum_{j=0}^{\infty} (\theta_r^j + \theta_r^{j\alpha}) =: V < \infty. \end{aligned} \quad \square$$

6.4. Expansivity. We want to obtain a bound on h_g^* , the tail entropy of g . By results of [21], the tail entropy may be positive. We assume that β is chosen not too large so that $(1 + \beta)^2 < 2$.

Lemma 6.10. *Let $\delta \in (0, 6\eta)$. Given $n \in \mathbb{N}$, and $x, z \in \mathbb{T}^4$ such that $d_n(x, z) < 6\eta$, we have*

$$(6.5) \quad \Lambda_n^{\text{span}}(W_{6\eta}^{cu}(z) \cap B_n(x, 6\eta), 0, \delta; g) \leq 32(6\eta)^2 \delta^{-2} \lambda^{2n}.$$

Proof. Write $\varepsilon = 6\eta$. First we prove that

$$W_\varepsilon^{cu}(z) \cap B_k(x, \varepsilon) \subset g^{-(k-1)}(W_{4\varepsilon}^{cu}(g^{k-1}z))$$

for $k \in \{1, \dots, n\}$. This follows by induction; it is true for $k = 1$, and given the result for $k \in \{1, \dots, n-1\}$, we see that any $z' \in W_\varepsilon^{cu}(z) \cap B_{k+1}(x, \varepsilon)$ has $g^{k-1}(z') \in W_{4\varepsilon}^{cu}(g^{k-1}z)$ by the inductive hypothesis, and so

$$g^k(z') \in W_{4\varepsilon\|Dg\|}^{cu}(g^k z).$$

Also $g^k(z') \in B(g^k x, \varepsilon) \subset B(g^k z, 2\varepsilon)$, where the last inclusion follows because $4\varepsilon\|Dg\|$ is not enough distance to wrap all the way around the torus and enter $B(g^k x, \varepsilon)$ again. This is true because ε is assumed to be not too large. This is the only requirement on ε in this proof. Thus, by Lemma 3.7. $g^k(z') \in W_{2\varepsilon(1+\beta)^2}^{cu}(g^k z) \subset W_{4\varepsilon}^{cu}(g^k z)$. Now fix $\alpha = \delta(1 + \beta)^{-1} \lambda^{-n}$. Recall that $W_{4\varepsilon}^{cu}(g^n z)$ is the graph of a function from F^{cu} to F^{cs} with norm less than β . The projection of $W_{4\varepsilon}^{cu}(g^n z)$ to F^{cu} along F^{cs} is contained in a ball of radius 4ε , so $B_{4\varepsilon}(0)$ in F^{cu} has an α -dense subset in the d_n -metric with cardinality less than or

equal to $16\varepsilon^2\alpha^{-2}$. Projecting this set back to $W_{4\varepsilon}^{cu}(g^n z)$ along F^{cu} gives $E \subset W_{4\varepsilon}^{cu}(g^n z)$ that is $(1 + \beta)\alpha$ -dense.

Consider the set $g^{-n}(E) \subset W^{cu}(z)$. Given any $y \in W_\varepsilon^{cu}(z) \cap B_n(x, \varepsilon)$, we have $g^n(y) \in W_{4\varepsilon}^{cu}(g^n z)$ and so there is $z' \in E$ such that $d_{cu}(g^n y, g^n z') < (1 + \beta)\alpha$. Since g^{-1} expands distances along W^{cu} by at most λ , we have $d_n(y, z') < (1 + \beta)\alpha\lambda^n$. We see that $g^{-n}(E)$ is an (n, δ) -spanning set for $W_\varepsilon^{cu}(z) \cap B_n(x, \varepsilon)$, and moreover

$$\#g^{-n}(E) \leq 16\varepsilon^2\alpha^{-2} \leq 16\varepsilon^2\delta^{-2}(1 + \beta)^2\lambda^{2n},$$

which gives (6.5) and completes the proof of Lemma 6.10. \square

Lemma 6.11. *For every $g \in \mathcal{V}$ we have $h_g^*(6\eta) \leq 6 \log \lambda$.*

Proof. Given $x \in \mathbb{T}^4$ and $\delta > 0$, we estimate $\Lambda_n^{\text{span}}(\Gamma_{6\eta}(x), 0, 2\delta; g)$ for $n \in \mathbb{N}$. To do this, we start by fixing

$$(6.6) \quad \alpha = \alpha(n) = \frac{\delta}{\kappa\lambda^n}$$

where κ is from the local product structure. Let $E \subset \Gamma_{6\eta}(x)$ be an α -dense set with cardinality

$$\#E \leq (12\eta/\alpha)^4 = (12\eta)^4\kappa^4\lambda^{4n}\delta^{-4};$$

note that such a set exists because $\Gamma_{6\eta}(x)$ is contained in $x + [-6\eta, 6\eta]^4$.

Now we have $W_{\kappa\alpha}^{cu}(z) \subset W_{6\eta}^{cu}(z)$ for each $z \in E$, so by Lemma 6.10, there is an (n, δ) -spanning set E_z for $W_{\kappa\alpha}^{cu}(z) \cap B_n(x, 6\eta)$ with

$$\#E_z \leq 32(6\eta)^2\delta^{-2}\lambda^{2n}.$$

Let $E' = \bigcup_{z \in E} E_z$, then we have

$$\#E' \leq 32(12\eta)^6\delta^{-6}\kappa^4\lambda^{6n}.$$

We claim that E' is $(n, 2\delta)$ -spanning for $\Gamma_{6\eta}(x)$, which will complete the proof of Lemma 6.11. To see this, take any $y \in \Gamma_{6\eta}(x)$, and observe that because E is α -dense in $B(x, 6\eta)$, there is $z = z(y) \in E \cap B(y, \alpha)$. By the local product structure there is $\bar{z} = \bar{z}(y) \in W_{\kappa\alpha}^{cs}(y) \cap W_{\kappa\alpha}^{cu}(z)$. Notice that because distance expansion along W^{cu} is bounded above by λ for each iteration of g , we have

$$(6.7) \quad d_n(y, \bar{z}) < \kappa\alpha\lambda^n = \delta.$$

By our choice of E_z , there is $z' \in E_z$ such that $d_n(z', \bar{z}) < \delta$. Thus $d_n(y, z') < 2\delta$, as required. It follows that

$$\Lambda_n^{\text{span}}(\Gamma_{6\eta}(x), 0, 2\delta; g) \leq 32(12\eta)^6\delta^{-6}\kappa^4\lambda^{6n},$$

hence $h_g^*(6\eta) \leq 6 \log \lambda$, which proves Lemma 6.11. \square

Lemma 6.12. *For every $r > \gamma(g)$ and $\varepsilon = 300\rho'$, the diffeomorphism g satisfies Condition [E].*

Proof. Suppose $x \in \mathbb{T}^4$, $r > 0$, and $n_k, m_k \rightarrow \infty$ are such that

$$(6.8) \quad \frac{1}{n_k} S_{n_k}^g \chi(x) \geq r, \quad \frac{1}{m_k} S_{m_k}^{g^{-1}} \chi'(x) \geq r$$

for every k . Our goal is to show that $\Gamma_\varepsilon(x) = \{x\}$.

First we fix $r' \in (\gamma, r)$ and observe that by Pliss' lemma [34] there are $m'_k, n'_k \rightarrow \infty$ such that

$$(6.9) \quad \begin{aligned} S_m^g \chi'(g^{-m'_k} x) &\geq mr' \text{ for every } 0 \leq m \leq m'_k, \\ S_n^{g^{-1}} \chi(g^{n'_k} x) &\geq nr' \text{ for every } 0 \leq n \leq n'_k. \end{aligned}$$

As in the proof of Lemma 6.6, for every $y \in B_{m'_k}(g^{-m'_k} x, \rho'')$ and $z \in g^{n'_k} B_{n'_k}(x, \rho'')$, we now have

$$(6.10) \quad \begin{aligned} \|Dg^m(y)|_{E^{cs}}\| &\leq \theta_{r'}^m \text{ for every } 0 \leq m \leq m'_k, \\ \|Dg^{-n}(z)|_{E^{cu}}\| &\leq \theta_{r'}^n \text{ for every } 0 \leq n \leq n'_k, \end{aligned}$$

where $\theta_{r'} < 1$ is as in (4.4).

Now let $x' \in \Gamma_\varepsilon(x)$. By the local product structure, and ε being not too large, there is a unique point $x'' \in W_{\kappa\varepsilon}^{cu}(x) \cap W_{\kappa\varepsilon}^{cs}(x')$. Applying g we see that

$$g(x'') \in W_{\kappa\varepsilon\|Dg\|}^{cs}(gx) \cap W_{\kappa\varepsilon\|Dg^{-1}\|}^{cu}(gx').$$

But by the local product structure, $W_{\kappa\varepsilon\|Dg\|}^{cs}(gx)$ and $W_{\kappa\varepsilon\|Dg^{-1}\|}^{cu}(gx')$ have a unique intersection point if $\max\{\kappa\varepsilon\|Dg\|, \kappa\varepsilon\|Dg^{-1}\|\} < 6\eta$. Thus $g(x'')$ is the unique intersection point, and since $d(gx, gx') \leq \varepsilon$, it follows that $g(x'') \in W_{\kappa\varepsilon}^{cs}(gx) \cap W_{\kappa\varepsilon}^{cu}(gx')$. Iterating the above argument gives for every $n \in \mathbb{Z}$,

$$(6.11) \quad g^n(x'') \in W_{\kappa\varepsilon}^{cu}(g^n x) \cap W_{\kappa\varepsilon}^{cs}(g^n x').$$

In particular, for each $k \in \mathbb{N}$ we can apply (6.10) with z a point along the W^{cu} -geodesic from $g^{n'_k} x$ to $g^{n'_k} x''$, and deduce that

$$d_{cu}(x, x'') \leq \theta_{r'}^{n'_k} d_{cu}(g^{n'_k} x, g^{n'_k} x'') \leq \theta_{r'}^{n'_k} \kappa\varepsilon.$$

Sending $k \rightarrow \infty$ gives $d_{cu}(x, x'') = 0$ and hence $x'' = x$ since $x'' \in W_{\kappa\varepsilon}^{cu}(x)$. Now by (6.11) we have $g^n x \in W_{\kappa\varepsilon}^{cs}(g^n x')$ for all $n \in \mathbb{Z}$, and for each $k \in \mathbb{N}$ we can apply (6.10) with y a point along the W^{cs} -geodesic from $g^{-m'_k} x$ to $g^{-m'_k} x'$, obtaining

$$d_{cs}(x, x') \leq \theta_{r'}^{m'_k} d_{cs}(g^{-m'_k} x, g^{-m'_k} x') \leq \theta_{r'}^{m'_k} \kappa\varepsilon.$$

Again, as k increases we get $d_{cs}(x, x') = 0$ hence $x' = x$, which completes the proof of Lemma 6.12. \square

6.5. Verification of Theorem 4.1. We now have all the ingredients to show that if $g \in \mathcal{V}(f_{BV})$ and $\varphi: \mathbb{T}^4 \rightarrow \mathbb{R}$ satisfy the hypotheses of Theorem 4.1, then the conditions of Theorem 2.8 are satisfied, and hence there is a unique equilibrium state for $(\mathbb{T}^4, g, \varphi)$.

We define the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ as in Lemma 4.4. In Lemma 6.7, we showed that \mathcal{G}^M has tail specification at scale $3\rho'$, so condition (1) of Theorem 2.8 holds. In Lemma 6.9, we showed that φ has the Bowen property on \mathcal{G} at scale $300\rho'$, so condition (2) of Theorem 2.8 holds. We have $P(\mathcal{P} \cup \mathcal{S}, \varphi, 6\eta) = \max\{P(\mathcal{P}, \varphi, 6\eta), P(\mathcal{S}, \varphi, 6\eta)\}$ and both collections satisfy the hypotheses of Theorem 3.3, and thus we have the upper bound

$$(1-r) \sup_{x \in Q} \varphi(x) + r(\sup_{x \in \mathbb{T}^4} \varphi(x) + h + \log L) + H(2r),$$

and r can be chosen arbitrarily close to γ . By Lemma 6.11, $h_g^*(6\eta) < 6 \log \lambda$, so by Theorem 3.3, $P(\mathcal{P} \cup \mathcal{S}, \varphi)$ is bounded above by

$$6 \log \lambda + (1-r) \sup_{x \in Q} \varphi(x) + r(\sup_{x \in \mathbb{T}^4} \varphi(x) + h + \log L) + H(2r).$$

Thus, the hypothesis of Theorem 4.1 gives that

$$P(\mathcal{P} \cup \mathcal{S}, \varphi) + \text{Var}(\varphi, 300\rho') < P(\varphi; g),$$

which verifies condition (3) of Theorem 2.8. Finally, by Theorem 3.4 and Lemma 6.12, we have $P_{\text{exp}}^\perp(\varphi, 300\rho') \leq P(\mathcal{P} \cup \mathcal{S}, \varphi) < P(\varphi; g)$.

Combining these ingredients, we see that under the conditions of Theorem 4.1, all the hypotheses of Theorem 2.8 are satisfied for the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$. This completes the proof of Theorem 4.1.

7. SRB MEASURES AND PROOF OF THEOREM B

An *SRB measure* for a C^2 diffeomorphism f is an ergodic invariant measure μ that is hyperbolic (non-zero Lyapunov exponents) and has absolutely continuous conditional measures on unstable manifolds [1, Chapter 13]. We assume that g is a C^2 diffeomorphism in a C^1 neighborhood of a Bonatti–Viana diffeomorphism $f_{BV} \in \mathcal{F}_{\lambda, \rho}$ with $\log \lambda$ and ρ not too large. Explicit bounds required on the parameters for f_{BV} are given at (7.4).

7.1. Geometric potential. We call $\varphi^{\text{geo}}(x) := -\log \det(Dg|_{cu}(x))$ the geometric potential because we will see that it is the potential function for which the SRB measure is a unique equilibrium state. It is a folklore result that a C^2 diffeomorphism with a dominated splitting has Hölder continuous distributions. However, to the best of our

knowledge a proof has never appeared in the literature. For diffeomorphisms of surfaces, this result is given in [35]. The idea of proof for the general result is to modify the C^r section theorem from Hirsch, Pugh and Shub [22]. Before we were informed of the folklore result, we worked out a direct proof that φ^{geo} has the Bowen property on \mathcal{G} , without showing it is Hölder. That argument was included in a preprint version of this paper¹ but is now omitted since the folklore result makes it unnecessary.

7.2. Non-negativity of pressure. We prove a general result on non-negativity of pressure for the geometric potential associated to an invariant foliation. Let M be a compact Riemannian manifold and W be a C^0 foliation of M with C^1 leaves. Suppose there is $\delta > 0$ such that

$$(7.1) \quad \sup_{x \in M} m_{W(x)}(W_\delta(x)) < \infty,$$

where $m_{W(x)}$ denotes volume on the leaf $W(x)$ with the induced metric.

Lemma 7.1. *Let W be a foliation of M as above, with $\delta > 0$ such that (7.1) holds. Let $f: M \rightarrow M$ be a diffeomorphism and let $\psi(x) = -\log |\det Df(x)|_{T_x W(x)}$. Then $P(\psi; f) \geq 0$.*

Proof. Note that ψ is continuous because f is C^1 and W is C^0 . Thus for every $\varepsilon > 0$, there is $\delta > 0$ such that $d(x, y) < \delta$ implies

$$(7.2) \quad |\psi(x) - \psi(y)| < \varepsilon.$$

Decreasing δ if necessary, we can assume that (7.1) holds. Now for every $x \in M$ and every $y \in B_n(x, \delta)$, we have

$$(7.3) \quad |\det Df^n(y)|_{T_y W(y)} \geq e^{-\varepsilon n} e^{-S_n \psi(x)}.$$

Writing $B_n^W(x, \delta)$ for the connected component of $W(x) \cap B_n(x, \delta)$ containing x , we get

$$m_{W(f^n x)}(f^n B_n^W(x, \delta)) \geq e^{-\varepsilon n} e^{-S_n \psi(x)} m_{W(x)} B_n^W(x, \delta).$$

Since $f^n B_n^W(x, \delta) \subset W_\delta(f^n x)$, we write C for the quantity in (7.1) and get $m_{W(x)} B_n^W(x, \delta) \leq C e^{\varepsilon n} e^{S_n \psi(x)}$ for every x, n .

Now let V be a local leaf of W . Given $n \in \mathbb{N}$, let Z_n be a maximal (n, δ) -separated subset of V . Then $V \subset \bigcup_{x \in Z_n} B_n^W(x, \delta)$, and so

$$m_V(V) \leq \sum_{x \in Z_n} m_V B_n^W(x, \delta) \leq \sum_{x \in Z_n} C e^{\varepsilon n} e^{S_n \psi(x)} \leq C e^{\varepsilon n} \Lambda_n^{\text{sep}}(\psi, \delta).$$

We conclude that $P(\psi; f) \geq P(\psi, \delta; f) \geq -\varepsilon$, and since $\varepsilon > 0$ was arbitrary this shows that $P(\psi; f) \geq 0$. \square

¹Version 2 of arXiv:1505.06371

We claim that Property (7.1) holds for the center-unstable foliation W^{cu} of g . Indeed, each local leaf $W_\delta(x)$ is the graph of a function $\psi: F^u \rightarrow F^s$ with $\|D\psi\| \leq \beta$, and writing $W'_\delta(x) \subset F^u$ for the projection of $W_\delta(x)$ to F^u along F^s , we see that

- (1) $W_\delta(x) = (\text{Id} + \psi)(W'_\delta(x))$,
- (2) $W'_\delta(x)$ is contained inside a ball of radius $\delta(1 + \beta)$ in F^u , and
- (3) $m_{W(x)}W_\delta(x) \leq (1 + \|D\psi\|)m_{F^u}W'_\delta(x) \leq (1 + \beta)\pi(\delta(1 + \beta))^2$.

Thus, we conclude that $P(\varphi^{\text{geo}}; g) \geq 0$.

7.3. Negativity of $\Phi(\varphi^{\text{geo}}; g)$. We show that $\Phi(\varphi^{\text{geo}}; g) < 0$ as long as the parameters in the Bonatti–Viana construction are chosen small.

Observe that $\sup_{x \in \mathbb{T}^4} \varphi^{\text{geo}}(x) \approx \log \lambda - \log \lambda_4$ and $\inf_{x \in \mathbb{T}^4} \varphi^{\text{geo}}(x) \approx -(\log \lambda_3 + \log \lambda_4)$. More precisely, given $\varepsilon > 0$, we can choose g in a sufficiently small C^1 neighbourhood of f_{BV} so that $\sup_{x \in \mathbb{T}^4} \varphi^{\text{geo}}(x) \leq \log \lambda - \log \lambda_4 + \varepsilon$, and $\inf_{x \in \mathbb{T}^4} \varphi^{\text{geo}}(x) \geq -(\log \lambda_3 + \log \lambda_4) - \varepsilon$. Thus,

$$\begin{aligned} \sup \varphi^{\text{geo}} + \text{Var}(\varphi^{\text{geo}}, 300\rho') &\leq 2 \sup \varphi^{\text{geo}} - \inf \varphi^{\text{geo}} \\ &\leq 2 \log \lambda + \log \lambda_3 - \log \lambda_4 + 2\varepsilon. \end{aligned}$$

Thus, we have

$$\begin{aligned} \Phi(\varphi^{\text{geo}}; g) &\leq 6 \log \lambda + \sup \varphi^{\text{geo}} + \gamma(\log L + h) + H(\gamma) + V \\ &\leq (\log \lambda_3 - \log \lambda_4) + 8 \log \lambda + \gamma(\log L + h) + H(\gamma) + 2\varepsilon, \end{aligned}$$

where, since $\lambda_4 > \lambda_3 > 1$, the first term is a negative number, and the other terms can be made small. Thus, $\Phi(\varphi^{\text{geo}}; g) < 0$. To be more precise, if $\lambda(f_{BV})$ is chosen small enough so that

$$(7.4) \quad 8 \log \lambda + \gamma(\log L + h) + H(\gamma) < \log \lambda_4 - \log \lambda_3,$$

then a sufficiently small C^1 perturbation of f_{BV} satisfies $\Phi(\varphi_g^{\text{geo}}; g) < 0$.

Since $\Phi(\varphi^{\text{geo}}; g) < 0 \leq P(\varphi^{\text{geo}}; g)$, we can apply Theorem A, and we obtain that φ^{geo} has a unique equilibrium state.

7.4. Proof that $\Phi(t\varphi^{\text{geo}}; g) < P(t\varphi^{\text{geo}}; g)$ for $t \in [0, 1]$. We show that the pressure bound $\Phi(t\varphi^{\text{geo}}; g) < P(t\varphi^{\text{geo}}; g)$ for all $t \in [0, 1]$ as long as (7.4) holds. Since the equality is strict, it will persist for all t in a neighborhood of $[0, 1]$. We give linear bounds for $P(t\varphi^{\text{geo}}; g)$ and $\Phi(t\varphi^{\text{geo}}; g)$. First observe that, by the variational principle,

$$\begin{aligned} P(t\varphi^{\text{geo}}; g) &\geq h_{\text{top}}(g) + t \inf \varphi^{\text{geo}} \\ &\geq h_{\text{top}}(g) - t(\log \lambda_3 + \log \lambda_4 + \varepsilon) \end{aligned}$$

Since there is a semi-conjugacy between g and f_A , $h_{\text{top}}(g) \geq h_{\text{top}}(f_A) = \log \lambda_3 + \log \lambda_4$. Thus, letting $a_1 = \log \lambda_3 + \log \lambda_4$, and

$$l_1(t) = a_1 - t(a_1 + \varepsilon),$$

we have $P(t\varphi^{\text{geo}}; g) \geq l_1(t)$ and $l_1(t) \geq 0$ whenever $t \leq \frac{a_1}{a_1 + \varepsilon}$.

Now, for $\Phi(t\varphi^{\text{geo}}; g)$, the argument of §7.3 shows that

$$\Phi(t\varphi^{\text{geo}}; g) \leq t(\log \lambda_3 - \log \lambda_4 + 2\varepsilon) + 8 \log \lambda + \gamma(\log L + h) + H(\gamma).$$

Thus, letting $a_2 = \log \lambda_4 - \log \lambda_3$ and $r = 8 \log \lambda + \gamma(\log L + h) + H(\gamma)$, and

$$l_2(t) = r - t(a_2 - 2\varepsilon),$$

we have $\Phi(t\varphi^{\text{geo}}; g) \leq l_2(t)$, and the root of $l_2(t)$ is $t^* = \frac{r}{a_2 - 2\varepsilon}$. Now suppose that

$$(7.5) \quad \frac{r}{a_2 - 2\varepsilon} < \frac{a_1}{a_1 + \varepsilon},$$

and that $r < a_1$. This is clearly possible since r can be chosen small. These criteria hold for ε small if (7.4) holds for $\lambda = \lambda(f_{BV})$. Since $l_2(0) < l_1(0)$ and $l_2(t^*) = 0 < l_1(t^*)$, then for $t \in [0, t^*]$,

$$\Phi(t\varphi^{\text{geo}}; g) \leq l_2(t) < l_1(t) \leq P(t\varphi^{\text{geo}}).$$

For $t \in (t^*, 1]$, we have $\Phi(t\varphi^{\text{geo}}; g) \leq l_2(t) < 0 \leq P(\varphi^{\text{geo}}) \leq P(t\varphi^{\text{geo}})$. The last inequality holds because since $\sup \varphi^{\text{geo}} < 0$, the function $t \mapsto P(t\varphi^{\text{geo}})$ is decreasing.

We conclude that $\Phi(t\varphi^{\text{geo}}; g) < P(t\varphi^{\text{geo}}; g)$ for all $t \in [0, 1]$, and thus there exists $\varepsilon > 0$ so $\Phi(t\varphi^{\text{geo}}; g) < P(t\varphi^{\text{geo}}; g)$ for all $t \in [-\varepsilon, 1 + \varepsilon]$. We apply Theorem A to these potentials, and we obtain uniqueness of these equilibrium states, which proves (2) of Theorem B.

7.5. The formula $P(\varphi^{\text{geo}}; g) = 0$ and μ_1 as SRB measure. Given a C^2 diffeomorphism f on a d -dimensional manifold and $\mu \in \mathcal{M}_e(f)$, let $\lambda_1 < \dots < \lambda_s$ be the Lyapunov exponents of μ , and let d_i be the multiplicity of λ_i , so that $d_i = \dim E_i$, where for a Lyapunov regular point x for μ we have

$$E_i(x) = \{0\} \cup \{v \in T_x M : \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda_i\} \subset T_x M.$$

Let $k = k(\mu) = \max\{1 \leq i \leq s(\mu) : \lambda_i \leq 0\}$, and let $\lambda^+(\mu) = \sum_{i>k} d_i(\mu)\lambda_i(\mu)$ be the sum of the positive Lyapunov exponents, counted with multiplicity.

The Margulis–Ruelle inequality [1, Theorem 10.2.1] gives $h_\mu(f) \leq \lambda^+(\mu)$, and it was shown by Ledrappier and Young [25] that equality holds if and only if μ has absolutely continuous conditionals on unstable manifolds. Thus, for any ergodic invariant measure μ , we have

$$(7.6) \quad h_\mu(f) - \lambda^+(\mu) \leq 0,$$

with equality if and only if μ is absolutely continuous on unstable manifolds. In conclusion, an ergodic measure μ is an SRB measure if and only if it is hyperbolic and equality holds in (7.6).

In this section, we prove that $P(\varphi^{\text{geo}}; g) \leq 0$. Combining this with Lemma 7.1 gives that $P(\varphi^{\text{geo}}; g) = 0$. To show that the unique equilibrium state μ for φ^{geo} is SRB, we need to show that μ is hyperbolic and $\lambda^+(\mu) = \int \varphi^{\text{geo}} d\mu$.

Lyapunov exponents for the diffeomorphism g . Let μ be ergodic, and let $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \lambda_3(\mu) \leq \lambda_4(\mu)$ be the Lyapunov exponents for μ . Recall that $E^{cs} \oplus E^{cu}$ is Dg -invariant, so for every μ -regular x the Oseledets decomposition is a sub-splitting of $E^{cs} \oplus E^{cu}$.

Lemma 7.2. *For an ergodic measure μ , then*

$$(7.7) \quad \int \varphi^{\text{geo}} d\mu \geq -\lambda^+(\mu).$$

Proof. Because $E^{cs} \oplus E^{cu}$ is dominated, standard arguments show that $\int \varphi^{\text{geo}} d\mu = -\lambda_3(\mu) - \lambda_4(\mu)$. There are three cases.

- (1) If μ has exactly two positive Lyapunov exponents (counted with multiplicity), then $\int \varphi^{\text{geo}} d\mu = -\lambda^+(\mu)$.
- (2) If $\lambda_2(\mu) \geq 0$, then $\int \varphi^{\text{geo}} d\mu \geq -\lambda_2(\mu) - \lambda_3(\mu) - \lambda_4(\mu) \geq -\lambda^+(\mu)$.
- (3) There is at most one positive Lyapunov exponent. In this case, $-\lambda_3 \geq 0$, so $\int \varphi^{\text{geo}} d\mu \geq -\lambda_4(\mu) \geq -\lambda^+(\mu)$. \square

Let $\mathcal{M}_* \subset \mathcal{M}_e(g)$ be the set of ergodic μ such that μ is hyperbolic and has exactly two positive exponents, so $\lambda_2(\mu) < 0 < \lambda_3(\mu)$.

Lemma 7.3. *If $\mu \in \mathcal{M}_e(g) \setminus \mathcal{M}_*$, then*

$$h_\mu(g) - \lambda^+(\mu) \leq h_\mu(g) + \int \varphi^{\text{geo}} d\mu \leq \Phi(\varphi^{\text{geo}}; g)$$

Proof. The first inequality follows from Lemma 7.2, so our work is to prove the second. Suppose that $\mu \in \mathcal{M}_e(g) \setminus \mathcal{M}_*$, and that either μ belongs to Case (1) and is not hyperbolic, or belongs to Case (2) above. Then there exists a set $Z \subset M$ with $\mu(Z) = 1$ so that for each $z \in Z$, there exists $v \in E_z^{cs}$ with $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Dg_z^n(v)\| \geq 0$. Thus with $r > \gamma$, we have $z \in A^+$, where

$$A^+ = \{x : \text{there exists } K(x) \text{ so } \frac{1}{n} S_n^g \chi(x) < r \text{ for all } n > K(x)\}.$$

To see this, suppose that $z \notin A^+$. Then there exists $n_k \rightarrow \infty$ with $\frac{1}{n_k} S_{n_k}^g \chi(z) \geq \gamma$. By Lemma 6.6, this gives

$$\|Dg_z^{n_k}(v)\| \leq \|Dg^{n_k}|_{E^{cs}(z)}\| \leq (\theta_r)^{n_k},$$

and thus $\lim_{n_k \rightarrow \infty} \frac{1}{n_k} \log \|Dg_z^{n_k}(v)\| \leq \log \theta_r < 0$, which is a contradiction. Thus, $\mu(A^+) = 1$. It follows that

$$h_\mu(g) - \lambda^+(\mu) \leq h_\mu(g) + \int \varphi^{\text{geo}} d\mu \leq P(\mathcal{C}, \varphi^{\text{geo}}) \leq \Phi(\varphi^{\text{geo}}; g),$$

where the first inequality uses (7.7), the second uses Theorem 3.5, and the third uses Theorem 3.3.

Now suppose μ belongs to case (3) above, and thus there is a non-positive exponent associated to E^{cu} . An analogous argument shows that $\mu(A^-) > 0$, where

$$A^- = \{x : \text{there exists } K(x) \text{ so } \frac{1}{n} S_n^{g^{-1}} \chi(x) < r \text{ for all } n > K(x)\}.$$

The key point is that there exists a set $Z \subset M$ with $\mu(Z) = 1$ so that for each $z \in Z$, there exists $v \in E_z^{cu}$ with

$$\lim_{n \rightarrow -\infty} \frac{1}{n} \log \|Dg_z^{-n}(v)\| \geq 0.$$

It follows that $z \in A^-$, because otherwise there exists $n_k \rightarrow \infty$ with $\frac{1}{n_k} S_{n_k}^{g^{-1}} \chi(z) \geq \gamma$, and thus by lemma 6.6, we have

$$\|Dg_z^{-n_k}(v)\| \leq \|Dg^{-n_k}|_{E^{cs}(z)}\| \leq (\theta_r)^{n_k},$$

and thus $\lim_{n_k \rightarrow -\infty} \frac{1}{n_k} \log \|Dg_z^{-n_k}(v)\| \leq \log \theta_r < 0$, which is a contradiction. Thus, $\mu(A^-) = 1$. Again, it follows that

$$h_\mu(g) - \lambda^+(\mu) \leq h_\mu(g) + \int \varphi^{\text{geo}} d\mu \leq P(\mathcal{C}, \varphi^{\text{geo}}) \leq \Phi(\varphi^{\text{geo}}; g).$$

where the first inequality uses (7.7), the second uses Theorem 3.5, and the third uses Theorem 3.3. \square

Completing the proof. It follows from §7.3, Lemma 7.3 and Lemma 7.1 that any ergodic μ not in \mathcal{M}_* satisfies

$$h_\mu(g) + \int \varphi^{\text{geo}} d\mu \leq \Phi(\varphi^{\text{geo}}) < 0 \leq P(\varphi^{\text{geo}}).$$

Thus, it follows from the variational principle that

$$(7.8) \quad P(\varphi^{\text{geo}}) = \sup \left\{ h_\mu(g) + \int \varphi^{\text{geo}} d\mu : \mu \in \mathcal{M}_* \right\}.$$

Now, for every $\mu \in \mathcal{M}_*$, we have $\int \varphi^{\text{geo}} d\mu = -\lambda^+(\mu)$, and thus

$$(7.9) \quad h_\mu(g) + \int \varphi^{\text{geo}} d\mu = h_\mu(g) - \lambda^+(\mu) \leq 0.$$

It follows that $P(\varphi^{\text{geo}}) = \sup \{h_\mu(g) + \int \varphi^{\text{geo}} d\mu : \mu \in \mathcal{M}_*\} \leq 0$. Hence, $P(\varphi^{\text{geo}}) = 0$. Since $\sup \varphi^{\text{geo}} < 0$, the function $t \mapsto P(t\varphi^{\text{geo}})$ is a convex strictly decreasing function from $\mathbb{R} \rightarrow \mathbb{R}$, and thus 0 is the unique root.

To show that the unique equilibrium state μ is an SRB measure for g , we observe that $\mu \in \mathcal{M}_*$ implies that μ is hyperbolic, and since $P(\varphi^{\text{geo}}) = 0$, (7.9) gives $h_\mu(g) - \lambda^+(\mu) = 0$, so μ is an SRB measure.

To see that there is no other SRB measure, we observe that if $\nu \neq \mu$ is any ergodic measure, then $h_\nu(g) - \lambda^+(\nu) \leq h_\nu(g) - \int \varphi^{\text{geo}} d\nu < P(\varphi^{\text{geo}}) = 0$ by (7.7) and the uniqueness of μ as an equilibrium measure. This completes the proof of Theorem B.

8. PROOFS OF LEMMAS

Proof of Lemma 2.1. It suffices to consider (n, δ) -separated sets of maximum cardinality in the supremum for the partition sum. Otherwise, we could increase the partition sum by adding in another point. An (n, δ) -separated set of maximum cardinality must be (n, δ) -spanning, or else we could add in another point and still be (n, δ) -separated. The first inequality follows.

For the second inequality, let E_n be any $(n, 2\delta)$ -separated set and F_n any (n, δ) -spanning set. Define the map $\pi: E_n \rightarrow F_n$ by choosing for each $x \in E_n$ a point $\pi(x)$ with the property that $d(x, \pi(x)) \leq \delta$. The map π is injective. Thus, for any E which is $(n, 2\delta)$ separated,

$$\sum_{y \in F_n} e^{S_n \varphi(y)} \geq \sum_{x \in E_n} e^{S_n \varphi(\pi(x))} \geq \sum_{x \in E_n} e^{S_n \varphi(x) - n \text{Var}(\varphi, \delta)},$$

and thus $\sum_{y \in F_n} e^{S_n \varphi(y)} \geq e^{-n \text{Var}(\varphi, \delta)} \Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, 2\delta)$. \square

Proof of Lemma 2.2. It is shown in [6, Proposition 2.2] that given any $\delta > 0$ and $\alpha > h_f^*(\varepsilon)$, there is a constant K such that

$$\Lambda_n^{\text{span}}(B_n(x, \varepsilon), 0, \delta; f) \leq K e^{\alpha n}$$

for every $x \in X$ and $n \in \mathbb{N}$; that is, every Bowen ball $B_n(x, \varepsilon)$ has an (n, δ) -spanning subset $F_{x,n}$ with cardinality at most $K e^{\alpha n}$. Let $E_n \subset \mathcal{D}_n$ be a maximal (n, ε) -separated set. Then $G_n = \bigcup_{x \in E_n} F_{x,n}$ is (n, δ) -spanning for \mathcal{D}_n , and has

$$\sum_{y \in G_n} e^{S_n \varphi(y)} \leq \sum_{x \in E_n} e^{S_n \varphi(x)} e^{n \text{Var}(\varphi, \varepsilon)} K e^{\alpha n}.$$

We conclude that $\Lambda_n^{\text{span}}(\mathcal{D}, \varphi, \delta) \leq \Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, \varepsilon) K e^{n(\text{Var}(\varphi, \varepsilon) + \alpha)}$. Then the second inequality in Lemma 2.1 gives

$$\Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, 2\delta) \leq e^{n \text{Var}(\varphi, \delta)} \Lambda_n^{\text{sep}}(\mathcal{D}, \varphi, \varepsilon) K e^{n(\text{Var}(\varphi, \varepsilon) + \alpha)},$$

sending $n \rightarrow \infty$ gives the first half of Lemma 2.2, and sending $\delta \rightarrow 0$ gives the second half. \square

Proof of Lemma 3.2. With η and C as in the statement of the lemma, put $\alpha = \eta/C$. By the Anosov shadowing lemma if $\{x_n\}$ is an α -pseudo orbit for f , then there exists an f -orbit that η -shadows $\{x_n\}$.

Now fix $g \in \text{Diff}(M)$ with $d_{C^0}(f, g) < \alpha$. Then every g -orbit is an α -pseudo orbit for f , and hence for every $x \in M$, we can find a unique point $\pi(x) \in M$ such that

$$(8.1) \quad d(f^n(\pi x), g^n x) < \eta \text{ for all } n \in \mathbb{Z}.$$

We prove (i). By expansivity of f , we have

$$(8.2) \quad P(\varphi; f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \Lambda_n^{\text{span}}(\varphi, 3\eta; f).$$

Let E_n be a (n, η) -spanning set for g . Then from (8.1) we see that $\pi(E_n)$ is $(n, 3\eta)$ -spanning for f . It follows that

$$(8.3) \quad \Lambda_n^{\text{span}}(\varphi, 2\eta; f) \leq \sum_{x \in \pi(E_n)} e^{S_n^f \varphi(x)} = \sum_{x \in E_n} e^{S_n^f \varphi(\pi x)}.$$

Note that $S_n^f \varphi(\pi x) = \sum_{k=0}^{n-1} \varphi(f^k(\pi x)) \leq \sum_{k=0}^{n-1} (\varphi(g^k x) + \text{Var}(\varphi, \eta))$, and together with (8.2) and (8.3) this gives

$$P(\varphi; f) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in E_n} e^{n \text{Var}(\varphi, \eta) + S_n^g \varphi(x)}.$$

Taking an infimum over all (n, η) -spanning sets for g gives

$$P(\varphi; f) \leq \text{Var}(\varphi, \eta) + P(\varphi, \eta; g)$$

by the first inequality in Lemma 2.1. This completes the proof of (i) since $P(\varphi; g) \geq P(\varphi, \eta; g)$.

Now we prove (ii). Let E_n be a maximal $(n, 3\eta)$ separated set for g . As in the previous argument, we see from (8.1) that $\pi(E_n)$ is (n, η) -separated for f : indeed, for every $x, y \in E_n$ there is $0 \leq k < n$ such that $d(g^k x, g^k y) \geq 3\eta$, and hence

$$d(f^k(\pi x), f^k(\pi y)) \geq d(g^k x, g^k y) - d(g^k x, f^k \pi x) - d(g^k y, f^k \pi y) > \eta.$$

In particular, we have

$$\begin{aligned} \Lambda_n^{\text{sep}}(\varphi, \eta; f) &\geq \sum_{x \in \pi(E_n)} e^{S_n^f \varphi(x)} = \sum_{x \in E_n} e^{S_n^f \varphi(\pi x)} \\ &\geq \sum_{x \in E_n} e^{S_n^g \varphi(x) - n \text{Var}(\varphi, \eta)} \geq \Lambda_n^{\text{sep}}(\varphi, 3\eta; g) e^{-n \text{Var}(\varphi, \eta)}. \quad \square \end{aligned}$$

Proof of Lemma 3.6. Given $x, y \in F^1 \oplus F^2$, let z' be the unique point of intersection of $(x + F^1) \cap (y + F^2)$. Translating the coordinate system so that z' becomes the origin, we assume w.l.o.g. that $x \in F^1$ and $y \in F^2$. Then $W^1(x)$ and $W^2(y)$ are graphs of C^1 functions $\phi_1: F^1 \rightarrow F^2$ and

$\phi_2: F^2 \rightarrow F^1$ with $\|D\phi_i\| < \beta$. That is, $W^1(x) = \{a + \phi_1(a) : a \in F^1\}$ and $W^2(y) = \{\phi_2(b) + b : b \in F^2\}$. Thus $z \in W^1 \cap W^2$ if and only if $z = a + \phi_1(a) = \phi_2(b) + b$ for some $a \in F^1$ and $b \in F^2$. This occurs if and only if $b = \phi_1(a)$ and $a = \phi_2(b)$; that is, if and only if $a = \phi_2 \circ \phi_1(a)$ and $b = \phi_1(a)$. Because $\phi_2 \circ \phi_1$ is a contraction on the complete metric space F^1 it has a unique fixed point a .

For the estimate on the distances from z to x, y we observe that

$$\begin{aligned} \|a\| &= d(a, 0) = d(\phi_2 b, \phi_2 y) \leq \beta d(b, y) \leq \beta(\|b\| + \|y\|), \\ \|b\| &= d(b, 0) = d(\phi_1 a, \phi_1 x) \leq \beta d(a, x) \leq \beta(\|a\| + \|x\|). \end{aligned}$$

Recall that by the definition of $\bar{\kappa}$ we have $\|x\|, \|y\| \leq \bar{\kappa}\|x - y\|$. Thus we have

$$\|a\| \leq \beta(\beta(\|a\| + \|x\|) + \|y\|) \leq \beta^2\|a\| + \beta(1 + \beta)\bar{\kappa}d(x, y),$$

which gives $\|a\| \leq \frac{\beta}{1-\beta}\bar{\kappa}d(x, y)$, and similarly for $\|b\|$. Thus

$$d(a, x) \leq \|a\| + \|x\| \leq \left(\frac{\beta}{1-\beta} + 1\right)\bar{\kappa}d(x, y) = \frac{\bar{\kappa}d(x, y)}{1-\beta}.$$

To obtain the bound on $d_{W^1}(z, x)$, observe that there is a path γ from a to x with length $\leq \frac{\bar{\kappa}}{1-\beta}d(x, y)$; the image of γ under the map $\text{Id} + \phi_1$ connects z to x and has length $\leq \frac{1+\beta}{1-\beta}\bar{\kappa}d(x, y)$ since $\|\text{Id} + \phi_1\| \leq 1 + \beta$. The other distance bound is similar. \square

Proof of Lemma 3.7. That is, whenever x, y are on the same local leaf, we have $d(x, y) \leq d_W(x, y) \leq (1 + \beta)^2 d(x, y)$ for $W = W^1, W^2$. In the case $W = W^1$, the factor of $(1 + \beta)^2$ comes from projecting y along F^2 to $x + F^1$, to get y' which is at most $(1 + \beta)d(x, y)$ from x . Let γ be the geodesic along $x + F^1$ connecting x and y' . Take the image of γ under the map $x + F^1 \rightarrow W^1(x)$, whose norm is at most $1 + \beta$. \square

Proof of Lemma 6.2. We use the following general lemma.

Lemma 8.1. *Let W be a foliation of a compact manifold M such that $W(x)$ is dense in M for every $x \in M$. Then for every $\alpha > 0$ there is $R > 0$ such that $W_R(x)$ is α -dense in M for every $x \in M$.*

Proof. Given $R > 0$, define a function $\psi_R: M \times M \rightarrow [0, \infty)$ by $\psi_R(x, y) = \text{dist}(y, W_R(x))$. Note that for each R , the map $x \mapsto W_R(x)$ is continuous (in the Hausdorff metric) and hence ψ_R is continuous. Moreover, since $W(x) = \bigcup_{R>0} W_R(x)$ is dense in M for each $x \in M$, we have $\lim_{R \rightarrow \infty} \psi_R(x, y) = 0$ for each $x, y \in M$. Finally, when $R \geq R'$ we see that $W_R(x) \supset W_{R'}(x)$ and so $\psi_R(x, y) \leq \psi_{R'}(x, y)$. Thus $\{\psi_R : R > 0\}$ is a family of continuous functions that converge monotonically to 0 pointwise. By compactness of $M \times M$, the convergence

is uniform, hence for every $\alpha > 0$ there is R such that $\psi_R(x, y) < \alpha$ for all $x, y \in M$. \square

Now put $\delta = \rho'$. By the local product structure for W^{cs}, W^u we can put $\alpha = \delta/\kappa$ and observe that if $d(y, z) < \alpha$, then $W_\delta^u(z) \cap W_\delta^{cs}(y) \neq \emptyset$. By Lemma 8.1, there is $R > 0$ such that $W_R^u(x)$ is α -dense in \mathbb{T}^d for every $x \in \mathbb{T}^d$. Thus for every $x \in \mathbb{T}^d$ there is $z \in W_R^u(x)$ such that $d(y, z) < \alpha$, and thus $W_\delta^u(z) \cap W_\delta^{cs}(y) \neq \emptyset$. The result follows by observing that $W_{R+\delta}^u(x) \supset W_\delta^u(z)$. \square

Acknowledgments. We would like to thank Keith Burns, Martin Sambarino, and Amie Wilkinson for helpful conversations. We also thank the American Institute of Mathematics, where some of this work was completed as part of a SQuaRE.

REFERENCES

- [1] L. Barreira and Y. Pesin. *Nonuniform hyperbolicity*, volume 115 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007. Dynamics of systems with nonzero Lyapunov exponents.
- [2] K. Berg. Convolution of invariant measures, maximal entropy. *Math. Systems Theory*, 3:146–150, 1969.
- [3] C. Bonatti, L. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005.
- [4] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.
- [5] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. revised edition.
- [6] R. Bowen. Entropy-expansive maps. *Trans. Amer. Math. Soc.*, 164:323–331, 1972.
- [7] R. Bowen. Some systems with unique equilibrium states. *Math. Systems Theory*, 8(3):193–202, 1974/75.
- [8] M. Boyle and T. Downarowicz. The entropy theory of symbolic extensions. *Invent. Math.*, 156(1):119–161, 2004.
- [9] H. Bruin and M. Todd. Equilibrium states for interval maps: the potential $-t \log |Df|$. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(4):559–600, 2009.
- [10] D. Burguet. A direct proof of the tail variational principle and its extension to maps. *Ergodic Theory Dynam. Systems*, 29(2):357–369, 2009.
- [11] J. Buzzi, S. Crovisier, and O. Sarig. Finite multiplicity for measures of maximal entropy of surface diffeomorphisms. Preprint, 2016.
- [12] J. Buzzi and T. Fisher. Entropic stability beyond partial hyperbolicity. *J. Mod. Dyn.*, 7(4):527–552, 2013.
- [13] J. Buzzi, T. Fisher, M. Sambarino, and C. Vásquez. Maximal entropy measures for certain partially hyperbolic, derived from Anosov systems. *Ergodic Theory Dynam. Systems*, 32(1):63–79, 2012.

- [14] J. Buzzi and O. Sarig. Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps. *Ergodic Theory Dynam. Systems*, 23(5):1383–1400, 2003.
- [15] M. Carvalho. Sinai -Ruelle-Bowen measures for N -dimensional derived from Anosov diffeomorphisms. *Ergodic Theory Dynam. Systems*, 13(1):21–44, 1993.
- [16] V. Climenhaga. Specification and towers in shift spaces. Preprint, arXiv:1502.00931, 2015.
- [17] V. Climenhaga, T. Fisher, and D. J. Thompson. Equilibrium states for Mañé diffeomorphisms. *arXiv:1703.05722*, to appear in *Ergodic Theory and Dynamical Systems*.
- [18] V. Climenhaga and D. J. Thompson. Intrinsic ergodicity beyond specification: β -shifts, S -gap shifts, and their factors. *Israel J. Math.*, 192(2):785–817, 2012.
- [19] V. Climenhaga and D. J. Thompson. Unique equilibrium states for flows and homeomorphisms with non-uniform structure. *Adv. Math.*, 303:745–799, 2016.
- [20] M. Denker, G. Keller, and M. Urbański. On the uniqueness of equilibrium states for piecewise monotone mappings. *Studia Math.*, 97(1):27–36, 1990.
- [21] L. Díaz and T. Fisher. Symbolic extensions and partially hyperbolic diffeomorphisms. *Discrete Contin. Dyn. Syst.*, 29(4):1419–1441, 2011.
- [22] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.
- [23] F. Hofbauer. On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. *Israel J. Math.*, 34(3):213–237 (1980), 1979.
- [24] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995.
- [25] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin’s entropy formula. *Ann. of Math. (2)*, 122(3):509–539, 1985.
- [26] R. Leplaideur, K. Oliveira, and I. Rios. Equilibrium states for partially hyperbolic horseshoes. *Ergodic Theory Dynam. Systems*, 31(1):179–195, 2011.
- [27] R. Mañé. Contributions to the stability conjecture. *Topology*, 17(4):383–396, 1978.
- [28] L. Mendoza. Ergodic attractors for diffeomorphisms of surfaces. *J. London Math. Soc. (2)*, 37(2):362–374, 1988.
- [29] M. Misiurewicz. Topological conditional entropy. *Studia Math.*, 55(2):175–200, 1976.
- [30] William Parry and Mark Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, (187-188):268, 1990.
- [31] F. Paulin, M. Pollicott, and B. Schapira. Equilibrium states in negative curvature. *Astérisque*, (373):viii+281, 2015.
- [32] Y. Pesin. *Dimension theory in dynamical systems*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1997.
- [33] S. Pilyugin. *Shadowing in dynamical systems*, volume 1706 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.
- [34] V. A. Pliss. On a conjecture of Smale. *Differencial'nye Uravnenija*, 8:268–282, 1972.
- [35] E. Pujals and M. Sambarino. On the dynamics of dominated splitting. *Ann. of Math. (2)*, 169(3):675–739, 2009.

- [36] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi, and R. Ures. Maximizing measures for partially hyperbolic systems with compact center leaves. *Ergodic Theory Dynam. Systems*, 32(2):825–839, 2012.
- [37] D. Ruelle. A measure associated with axiom-A attractors. *Amer. J. Math.*, 98(3):619–654, 1976.
- [38] M. Sambarino. A (short) survey on dominated splittings. In *Mathematical Congress of the Americas*, volume 656 of *Contemp. Math.*, pages 149–183. Amer. Math. Soc., Providence, RI, 2016.
- [39] O. Sarig. Phase transitions for countable Markov shifts. *Comm. Math. Phys.*, 217(3):555–577, 2001.
- [40] O. Sarig. Symbolic dynamics for surface diffeomorphisms with positive entropy. *J. Amer. Math. Soc.*, 26(2):341–426, 2013.
- [41] Ja. G. Sinai. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
- [42] R. Spatzier and D. Visscher. Equilibrium measures for certain isometric extensions of anosov systems. To appear in *Ergodic Theory and Dynamical Systems*, 2016.
- [43] R. Ures. Intrinsic ergodicity of partially hyperbolic diffeomorphisms with a hyperbolic linear part. *Proc. Amer. Math. Soc.*, 140(6):1973–1985, 2012.
- [44] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.
- [45] Lai-Sang Young. Large deviations in dynamical systems. *Trans. Amer. Math. Soc.*, 318(2):525–543, 1990.
- [46] Y. Pesin, S. Senti, and K. Zhang. Thermodynamics of towers of hyperbolic type. To appear in *Trans. Amer. Math. Soc.*, arXiv:1403.2989, 2014.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX
77204

E-mail address: `climenha@math.uh.edu`

DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UT
84602

E-mail address: `tfisher@mathematics.byu.edu`

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 100 MATH
TOWER, 231 WEST 18TH AVENUE, COLUMBUS, OHIO 43210

E-mail address: `thompson@math.osu.edu`