Entropy as a dimension

Vaughn Climenhaga

University of Houston

February 9, 2024

Suppose we are studying the growth of a sequence $a_n \nearrow \infty$.

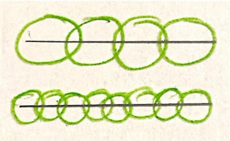
$a_n \approx \lambda^n$	$a_{n+1} \approx \lambda a_n$
$\log \lambda = \lim_{n\to\infty} \frac{1}{n} \log a_n$	$\lambda = \lim_{n\to\infty} a_{n+1}/a_n$
exists quite generally	existence is more restrictive
asymptotic growth rate	**scaling ratio**
box dimension	Hausdorff dimension
entropy?	entropy?

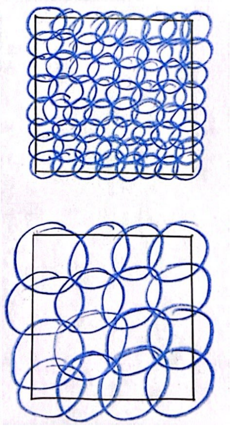
By adapting the "scaling ratio" approach of Hausdorff dimension to topological entropy, we can give elementary descriptions of the measure of maximal entropy and its product structure.

Dimension as an asymptotic growth rate

Using balls of radius r, it takes:

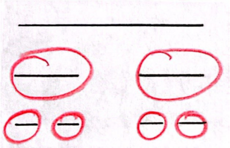
- $\approx \frac{1}{r}$ balls to cover the unit interval;
- $\approx \frac{1}{r^2}$ to cover the unit square.





$N(r)$ = number of balls $\approx 1/r^{\text{dimension}}$

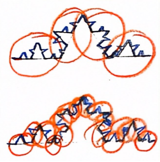
$\dim_B(X) = \lim_{r\to 0} \frac{\log N_X(r)}{\log(1/r)}$ is *box dimension*



Cantor set

$r = 3^{-k} \Rightarrow N(r) = 2^k$

$\dim_B = \frac{\log 2}{\log 3}$



Koch curve

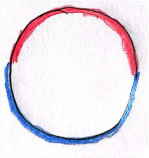
$N(r) = 4^k$

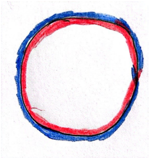
$\dim_B = \frac{\log 4}{\log 3}$

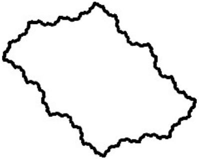
Dynamically significant sets: repellers and attractors

Consider the map $z \mapsto z^2$ in $\mathbb{C}$. Has unit circle S^1 as a repeller, $\dim_B = 1$.

For $c \approx 0$, the repeller (Julia set) of $z \mapsto z^2 + c$ is a quasicircle, $\dim_B > 1$.







From https://demonstrations.wolfram.com/QuadraticJuliaSets/

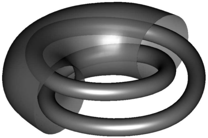
M a metric space, $f\colon M \to M$ continuous

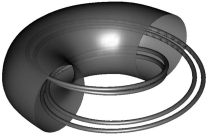
$U \subset M$ open, $\overline{f^{-1}(U)} \subset U$

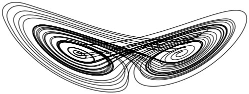
$$\Downarrow$$

repeller $X = \bigcap_{n=1}^{\infty} f^{-n}(U)$

If $\overline{f(U)} \subset U$, get attractor $X = \bigcap_{n=1}^{\infty} f^{n}(U)$. (Solenoid, Lorenz)



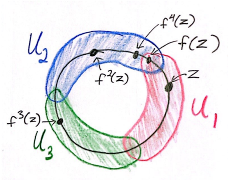




Topological entropy as an asymptotic growth rate

$h_{\text{top}}(X, f)$ = exponential growth rate of # of orbits of $f : X \to X$

- L_n = "# orbits of length n" $\approx e^{n h_{\text{top}}}$ $\Rightarrow$ $h_{\text{top}} = \lim \frac{1}{n} \log L_n$
- Preposterous because $L_n = \infty$. Need to "coarse-grain".



Adler
Konheim
McAndrew
(1965)

$\mathcal{U} = \{U_1, \ldots, U_d\}$ open cover

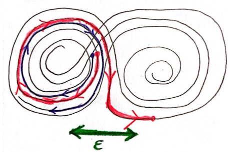
$w \in \{1, \ldots, d\}^n$ legal if $\exists\ z$ s.t. $z \in U_{w_1}$, $f(z) \in U_{w_2}$, etc.

- Picture $\Rightarrow$ 11232, 12232

L_n = # legal words of length n

$h_{\text{top}} = \sup_{\mathcal{U}} \overline{\lim}_{n\to\infty} \frac{1}{n} \log L_n$

Dinaburg
Bowen
(1971)



Count orbits up to scale $r > 0$

$\exists\ t \in [0, n]$ s.t. $d(f^t x, f^t y) \geq r$
$\Rightarrow x, y$ are (n, r)-separated

L_n = max #(n, r)-sep. set

$$h_{\text{top}} = \lim_{r\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log L_n$$

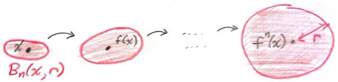
Entropy and box dimension

Bowen ball: $B_n(x,r) = \{y : d(f^t x, f^t y) < r \text{ for all } t \in [0,n]\}$

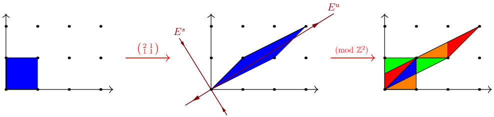
Box dim.	$B(x,r)$	r	$N(r) \approx r^{-d}$
Top. entropy	$B_n(x,r)$	e^{-n}	$L_n \approx e^{nh}$

$L_n = \#$ of Bowen balls to cover

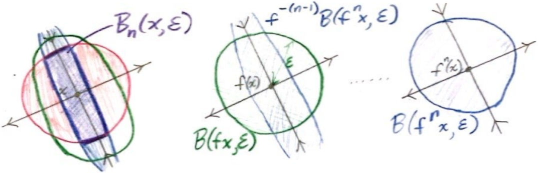
$$h_{\text{top}} = \lim_{r\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log L_n$$

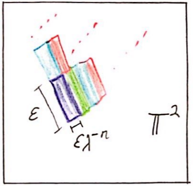


Expanding maps: some directions may refine more slowly



Hyperbolic maps: some directions don't refine at all!

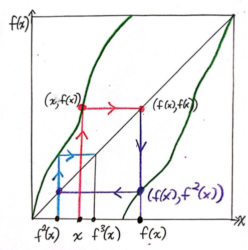


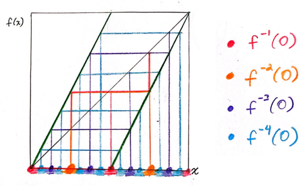


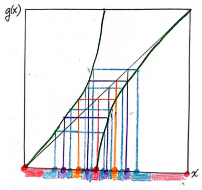
Preimages and periodic points

Piecewise expanding interval (circle) maps: simple geometry, rich dynamics

- Visualize orbits with cobweb diagram ($\updownarrow$ to graph, $\leftrightarrow$ to diagonal)
- Reverse direction to get preimages ($\updownarrow$ to diagonal, $\leftrightarrow$ to graph)







$\# f^{-n}(0) = 2^n = e^{nh}$

Do preimages equidistribute to Lebesgue? Something else?

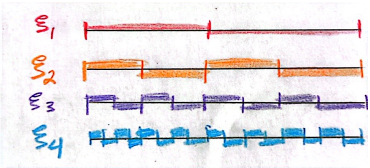
Each colored interval I has $f^4(I) = [0, 1]$, so applying Brouwer fixed point theorem to $f^4|_I^{-1}$ we get a 4-periodic point. In general $2^n = e^{nh}$ points of period n. Equidistribution?

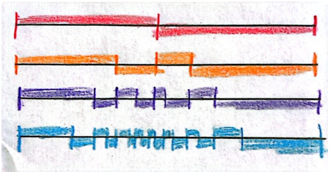
Measure-theoretic entropy as an asymptotic growth rate

Study $f\colon X \to X$ as a stochastic process using ergodic theory.

Given: f-invariant probability measure μ and a partition ξ

- $\xi_n = \bigvee_{k=0}^{n-1} f^{-k}\xi$, coarse-grain orbits of length n using ξ
- $H(\xi_n, \mu) = \sum_{C \in \xi_n} -\mu(C) \log \mu(C)$, expected information from making n observations in ξ
- Kolmogorov, Sinai (1950s): $\boxed{h(\mu) = \sup_\xi \lim \frac{1}{n} H(\xi_n, \mu)}$





Calculus: $H(\xi_n, \mu) \le \log \#\xi_n$, equality iff $\mu(C) = 1/\#\xi_n \ \forall \ C \in \xi_n$

The measure(s) of maximal entropy

X a compact metric space, $f: X \to X$ continuous

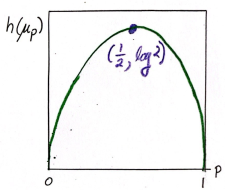
$$\mathcal{M}_f(X) = \{f\text{-invariant Borel probability measures on } X\}$$

This is a simplex, extreme points are ergodic measures. It is often infinite-dimensional with dense extreme points (Poulsen simplex).

Variational principle: $h_{\text{top}}(X, f) = \sup_\mu h(\mu)$

μ is a measure of maximal entropy (MME) if $h(\mu) = h_{\text{top}}(X, f)$.

Fact: both circle maps shown earlier have a unique MME. The preimage tree and the periodic orbits both equidistribute to it in a weak* sense.

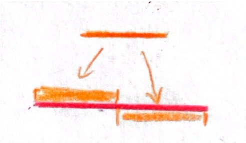
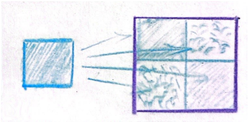
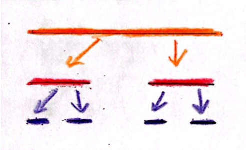


How general are these phenomena?

Dimension and entropy as a scaling ratio

Dimensional interpretation of $h(\mu)$? Dimension of a measure?

- Dimension of smallest set with full measure?
- Think of Lebesgue measure: scales like λ^d: $\mu(\lambda E) = \lambda^d \mu(E)$. Maybe we can generalize this?

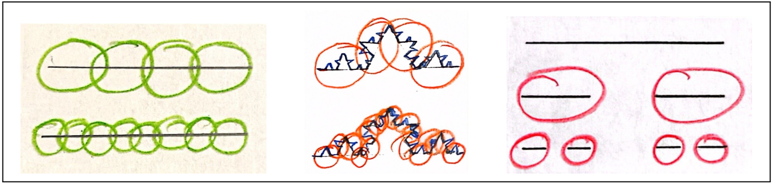
  

Does the MME have a nice scaling property like this?

> m is *conformal* if $\forall$ small E, we have $m(f(E)) = e^h m(E)$

How to construct a conformal measure?

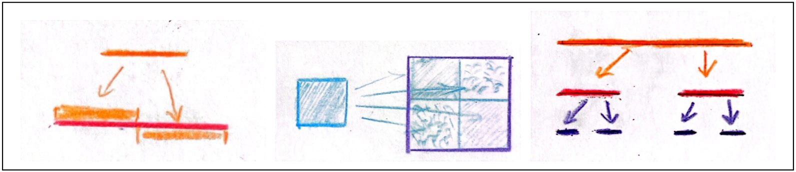
Shifting our viewpoint



dimension/entropy as an asymptotic growth rate

⇝ ⇝ ⇝

dimension/entropy as a scaling ratio (self-similarity)



Hausdorff dimension and measure

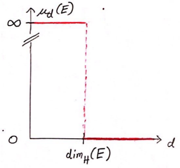
Lebesgue measure on $\mathbb{R}^d$ is (up to a constant)

$$\mu_d(E) = \lim_{r\to 0} \inf_{\{(x_i,r_i)\}} \left\{ \sum_i r_i^d : E \subset \bigcup_i B(x_i, r_i), r_i \le r \right\}$$

(If each $r_i = r$ then the sum is $N(r)r^d$...)

Same definition makes sense on any metric space X, with any $d \ge 0$ (not just $d \in \mathbb{N}$): gives *d-dimensional Hausdorff measure*.

Given $E \subset X$, graph of $d \mapsto \mu_d(E)$ is as shown.
Jump occurs at *Hausdorff dimension* $\dim_H(E)$.
Often get $\dim_H = \dim_B$, but not always.



Question: $0 < \mu_d(E) < \infty$? (sometimes yes, sometimes no)

Bowen, Pesin, Pitskel'

Bowen ball: $B_n(x,r) = \{y : d(f^t x, f^t y) < r \text{ for all } t \in [0,n]\}$

Dimension	$B(x,r)$	r	$N(r) \approx r^{-d}$	$\mu_d = \lim_r \inf \sum_i r_i^d$
Entropy	$B_n(x,r)$	e^{-n}	$L_n \approx e^{nh}$	$m_h = \lim_N \inf \sum_i e^{-n_i h}$

Bowen (1973): mimic Hausdorff measure using $B_n(x,r)$

$$m_h(E) = \lim_{N\to\infty} \inf_{\{(x_i,n_i)\}} \left\{ \sum_i e^{-n_i h} : E \subset \bigcup_i B_{n_i}(x_i,r), n_i \geq N \right\}$$

$$h_{\mathrm{top}}(E,r) = \sup\{h \geq 0 : m_h(E) = \infty\} = \inf\{h \geq 0 : m_h(E) = 0\}$$

$$h_{\mathrm{top}}(E) = \lim_{r\to 0} h_{\mathrm{top}}(E,r)$$

Agrees with previous definition if E is compact and f-invariant

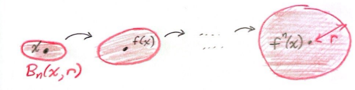
Pesin–Pitskel' (1984) extended to topological pressure

Pesin: theory of Carathéodory dimension characteristics

A conformal measure for expanding maps

Suppose $f: X \to X$ expanding (doubling map, quasicircle, etc.)

- Bowen balls refine to points $\Rightarrow m_h$ is a Borel measure
- m_h conformal: for small E, $m_h(f(E)) = e^h m_h(E)$



$$m_h(E) = \lim_N \inf_{\{(x_i,n_i)\}} \sum_i e^{-n_i h} \qquad (E \subset \bigcup_i B_{n_i}(x_i, r), n_i \geq N)$$

$$\{\text{covers of } E\} \leftrightarrow \{\text{covers of } f(E)\}$$

move by f, replace n_i by $n_i - 1$, scale weight by e^h

Two issues to deal with:

1. A priori, could have $m_h \equiv 0$ or $m_h(X) = \infty$.
2. m_h need not be invariant, so how do we get the MME?

Finiteness and invariance for expanding maps

Can guarantee $0 < m_h(X) < \infty$ as long as $C^{-1}e^{nh} \leq L_n \leq Ce^{nh}$

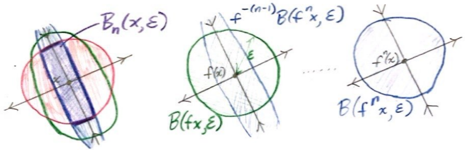
- C a constant, $L_n = \#$ of (n, r)-Bowen balls to cover X
- This kind of *uniform counting bound* can be proved for expanding maps using the specification property and an "almost-multiplicativity" argument. $(L_{n+k} = C^{\pm 1} L_n L_k)$

Going from conformal to invariant is a well-understood procedure (analogous to finding an absolutely continuous invariant measure w.r.t. Lebesgue). Two main techniques.

1. Pushforward and average: let $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_* m_h$ and $\mu = \lim_k \mu_{n_k}$, then μ is invariant, and can prove $\mu \ll m_h$.
2. Multiply by an appropriate density function: $d\mu = \psi \, dm_h$, where ψ is an eigenfunction for the Ruelle–Perron–Frobenius operator. (In fact, m_h is an eigenmeasure of the dual.)

Leaf measures in hyperbolic dynamics

Now consider uniformly hyperbolic $f\colon X \to X$, eg., solenoid or $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$



Bowen balls don't refine to points $\Rightarrow$ m_h is **not** a Borel measure on X

m_h **does** give a Borel measure m_x^u on local unstable leaf $W^u(x)$

- Reversing time gives a measure m_x^s on local stable leaf $W^s(x)$
- These measures scale by factors of $e^{\pm h}$ under f
- Originally built by Margulis (1970) using other techniques
- For Anosov flows, Hamenstadt and Hasselblatt (1989) described m_x^u as Hausdorff measure for appropriate metric ($d(x,y) = e^{-t(x,y)}$ where $t(x,y)$ is time to separate by r).
- C.–Pesin–Zelerowicz (BAMS 2019): given Hölder $\varphi\colon X \to \mathbb{R}$, used dimensional approach to construct $m_x^{\varphi,u}$ such that $m_{f(x)}^{\varphi,u}(f(E)) = \int_E e^{\varphi(y)-P(\varphi)}\, dm_x^{\varphi,u}(y)$

Push forward and average

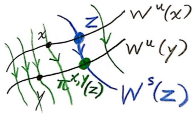
Theorem (C.–Pesin–Zelerowicz, 2019)

Let X be a transitive locally maximal hyperbolic set for a diffeomorphism f, and let $h = h_{\mathrm{top}}(X, f)$. Then:

1. *leaf measures $m_x^u(E) = \lim_N \inf \sum_i e^{-n_i h}$ are positive and finite;*
2. *they scale by $m_{f(x)}^u(f(E)) = e^h m_x^u(E)$;*
3. *they have absolutely continuous holonomies, $\pi_*^{x,y} m_x^u \ll m_y^u$;*
4. *for every $x \in X$, the measures $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m_x^u$ converge in the weak* topology to (a scalar multiple of) the unique MME.*

Parmenter and Pollicott (2022, arXiv) prove a version of (4) with $f_*^{-n}\mathrm{Leb}_{f^n(W_x^u)}$ replacing m_x^u.

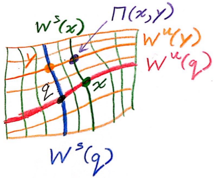
Conjecture: these measures converge to m_x^u.



These results extend to equilibrium measures for Hölder potentials.

A product construction

A set R is a *rectangle* if for all $x, y \in R$, the intersection $W^s(x) \cap W^u(y)$ is a single point, which itself lies in R.



Define $\Pi\colon W^u(q) \times W^s(q) \to R$ by $\Pi(x,y) = W^s(x) \cap W^u(y)$.

Definition

A measure μ has *product structure* if μ-a.e. x lies in a rectangle R where $\mu|_R$ is equivalent to $\Pi_*(m^u_x \times m^s_x)$ for some leaf measures.

Margulis: MME has $\mu|_R = \Pi_*(m^u_x \times m^s_x)$, and $\pi^{x,y}_* m^u_x = m^u_y$.
To achieve this with dimensional construction, tweak definition.
For uniformly hyperbolic flows it comes for free. This approach again extends to Hölder potentials, by introducing densities.
(Obtain a direct expression for *stable* conditionals of the SRB...)

[C., 2024, chapter in "A Vision for Dynamics in the 21st Century: The Legacy of Anatole Katok"]

A direct two-sided construction

An alternate approach is to use two-sided Bowen balls. For a uniformly hyperbolic attractor X of a smooth flow, define

$$B^*_{s,t}(x,r) = \left\{ y : \sup_{\tau \in [-s,t]} d(f^\tau x, f^\tau y) < r \text{ and } |\beta(x,y)| < \frac{r}{s+t} \right\}$$

Here $\beta(x,y)$ is "time lag" between $W^s(x)$ and $W^u(y)$.

Theorem (C., 2024, arXiv:2009.09260)

For $r > 0$ small, there exists $c > 0$ such that the unique MME is

$$\mu_{\mathrm{MME}}(E) = \lim_{T\to\infty} \inf \sum_i \frac{c}{s_i + t_i} e^{-(s_i+t_i) h_{\mathrm{top}}(X,f)},$$

where inf is over all $\{(x_i, s_i, t_i)\}_i$ such that $E \subset \bigcup_i B^*_{s_i,t_i}(x_i, r)$ and $s_i, t_i \geq T$. Similarly, the SRB measure is

$$\mu_{\mathrm{SRB}}(E) = \lim_{T\to\infty} \inf \sum_i \frac{c'}{s_i + t_i} \det(Df_{s+t}|_{E^u_{f^{-s}(x)}})^{-1}.$$

Further directions

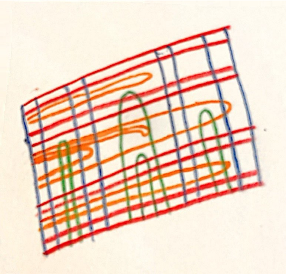
Mixing Anosov flows: Margulis proved $\text{Per}(T) \sim \frac{e^{hT}}{hT}$, where $\text{Per}(T)$ is the # of periodic orbits with length $\leq T$, and $\sim$ means $\frac{LHS}{RHS} \to 1$.

Key ingredients are product structure of MME, scaling properties of $m_x^{u,s}$. Can we follow Margulis argument in more general settings?

C.–Knieper–War, 2022: geodesics on surfaces w/o conjugate points

Beyond uniform hyperbolicity: product structure has "holes" (Pesin theory).

Idea: produce (Cantor) rectangle where $0 < m_x^{u,s} < \infty$, then push forward $m_x^u \times m_x^s$ and prove expected return time is finite.



Next: Jason will explain how we do this for Sinai billiards.