Thermodynamics for discontinuous maps and potentials

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July 11, 2013

Plan of talk

- Dynamical system $\begin{cases} X \text{ a complete separable metric space} \\ f\colon X \to X \text{ a measurable map} \end{cases}$
- Potential function $\varphi\colon X \to [-\infty, \infty]$ measurable

Classical thermodynamic formalism:

- Assume X compact, f continuous, φ continuous
- Relate two definitions of pressure: **supremum, growth rate**

Problem: Many interesting examples violate one or more of these

- Still get some version of the variational principle, using some aspect of the structure of the system

Goal: formulate a general statement valid for all systems

Trailer (with spoilers!)

Theorem

Let X be a complete separable metric space, $f: X \to X$ a measurable map, and $\varphi\colon X \to \mathbb{R}$ a bounded measurable function. Then $P(\varphi) = \sup\{h_\mu(f) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_f(X)\}$.

$P(\varphi) = \sup\{P(\mathcal{D}, \varphi) \mid \mathcal{D}$ is a topologically separated set of orbit segments satisfying **(C1)**$\}$

(C1) Uniform tightness: $\forall \epsilon > 0$ there is compact $Z_\epsilon \subset X$ s.t.

- $\mathcal{E}_{x,n}(X \setminus Z_\epsilon) < \epsilon$ for all large n and $x \in \mathcal{D}_n$
- $f|_{Z_\epsilon}$ and $\varphi|_{Z_\epsilon}$ are continuous

Basic notation and space of orbit segments

- X complete separable metric space, $f\colon X \to X$ measurable
- $\varphi\colon X \to [-\infty, \infty]$ a measurable potential function

Many definitions given in terms of $X \times \mathbb{N}$, space of orbit segments

- Identify (x, n) with $x \to f(x) \to f^2(x) \to \cdots \to f^{n-1}(x)$
- Given $\mathcal{D} \subset X \times \mathbb{N}$, write $\mathcal{D}_n = \{x \mid (x, n) \in \mathcal{D}\}$

Examples:

- Let $\mathcal{D}_n$ be a maximal (n, δ)-separated set, put $\mathcal{D} = \bigcup_n \mathcal{D}_n$
- Fix $I \subset \mathbb{R}$, put $\mathcal{D} = \{(x, n) \mid \frac{1}{n} S_n\varphi(x) \in I\}$
- Non-uniformly expanding: $\mathcal{D} = \{(x, n) \mid n \text{ a hyp. time for } x\}$
- NUH: Fix ℓ, let $\mathcal{D} = \{(x, n) \mid x, f^n(x) \in \Lambda_\ell \text{ (Pesin set)}\}$

First $\mathcal{D}$ is "topologically separated", others are not.

A proliferation of definitions

(X, f, φ) as before, $\mathcal{M}_f^\varphi(X) = \{\text{Borel prob. meas. with } \varphi \in L^1(\mu)\}$

Various ways to define topological pressure in "classical" case

1. **Supremum:** $P(\varphi) = \sup\{h_\mu(f) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_f^\varphi(X)\}$
2. **Growth rate:** $\begin{cases} \Lambda_n(\mathcal{D}, \varphi) = \sum_{x \in \mathcal{D}_n} e^{S_n\varphi(x)} \\ P(\mathcal{D}, \varphi) = \lim \frac{1}{n} \log \Lambda_n(\mathcal{D}, \varphi) \\ P(\varphi) = \sup / \inf / \lim P(\mathcal{D}, \varphi) \end{cases}$

   Mimics packing/box dimension, coarse spectrum
   (Includes definition as spectral radius of $\mathcal{L}_\varphi$)
3. **Critical exponent:** $\begin{cases} m_\varphi(\alpha) = \inf_{\mathcal{D}} \sum_{(x,n) \in \mathcal{D}} e^{-\alpha n + S_n\varphi(x)} \\ P(\varphi) = \inf\{\alpha \mid m_\varphi(\alpha) = 0\} \end{cases}$

   Mimics Hausdorff dimension, fine spectrum

Fundamental results in classical setting

Compact invariant sets, cts f and $\varphi \Rightarrow$ all three notions coincide.

Extra information on system yields results on existence, uniqueness, and statistical properties of equilibrium states.

Definition via growth rates plays key role.

- **Existence:** for expansive systems, there is $\mathcal{D} \subset X \times \mathbb{N}$ with $P(\varphi) = P(\mathcal{D}, \varphi)$. Build μ with $h_\mu(f) + \int \varphi \, d\mu = P(\mathcal{D}, \varphi)$ as limit of combination of empirical measures $\mathcal{E}_{x,n} = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x}$
- **Uniqueness:** use some structure (specification, Markov, etc.) to show that μ is Gibbs, ergodic, unique
- **Statistical properties:** spectral gap for RPF operator $\mathcal{L}_\varphi$

Noncompactness and discontinuity

What about non-compact X and discontinuous f and φ?

- Piecewise expanding maps (interval or otherwise): f is discontinuous, natural potential $\varphi = -\log|f'|$ is discontinuous
- Interval maps with critical points: $-\log|f'|$ has singularities
- Lyapunov exponents for non-uniformly hyperbolic systems: $\log\det(Df|_{E^u})$ has same regularity as E^u
- Shift spaces on a countable alphabet: X is non-compact
- Geodesic flow on non-compact manifold

Definitions in general case

Growth rate definition can be made precise in various ways. (Spanning sets, separated sets, covers.) Not clear which to use when compactness and continuity fail.

Supremum definition is unambiguous, can be taken as definition of pressure for any metric space X and measurable f, φ.

$$P^*(\varphi) = \sup \left\{ h_\mu(f) + \int \varphi \, d\mu \;\middle|\; \mu \in \mathcal{M}_f^\varphi(X) \right\}$$

Question: Can this quantity still be interpreted as a growth rate in the non-compact and discontinuous setting?

Remark: Same question for $\mathcal{M}_f^{\varphi-}(X) = \{\mu \mid \int \varphi \, d\mu > -\infty\}$.

Topological separation

$$\Lambda_n(\mathcal{D},\varphi) = \sum_{x\in\mathcal{D}_n} e^{S_n\varphi(x)} \quad \rightsquigarrow \quad P(\mathcal{D},\varphi) = \overline{\lim}\, \frac{1}{n} \log \Lambda_n(\mathcal{D},\varphi)$$

To get $\Lambda_n(\mathcal{D},\varphi) < \infty$, require $\mathcal{D}$ to be "coarse".

- maximal (n,δ)-separated set
- minimal (n,δ)-spanning set
- fix open cover $\mathcal{U}$ indexed by $I = \{1,\dots,d\}$, for each $w \in I^n$ let $U(w) = \{x \mid f^k x \in U_{w_k} \text{ for each } 0 \le k < n\}$, then choose $x(w) \in U(w)$ and take $\mathcal{D}_n = \bigcup_{w\in I^n} x(w)$
- take $\mathcal{D}_n$ maximal with the property that there is $\mathcal{U}$ such that $\#(U(w) \cap \mathcal{D}_n) \le 1$ for each $w \in I^n$

We use the last one. Let $\mathcal{D} \subset X \times \mathbb{N}$ be a collection of orbit segments. Call it topologically separated if this last property holds.

Main result (simplified form)

Define $P(\varphi) = \sup\{P(\mathcal{D}, \varphi) \mid \mathcal{D} \text{ top. sep. satisfying } \mathbf{(C1)}\}$

(C1) Uniform tightness: $\forall \epsilon > 0$ there is compact $Z_\epsilon \subset X$ s.t.

- $\mathcal{E}_{x,n}(X \setminus Z_\epsilon) < \epsilon$ for all large n and $x \in \mathcal{D}_n$
- $f|_{Z_\epsilon}$ and $\varphi|_{Z_\epsilon}$ are continuous

Theorem

Let X be a complete separable metric space, $f : X \to X$ a measurable map, and $\varphi \colon X \to \mathbb{R}$ a bounded measurable function. Then $P(\varphi) = P^*(\varphi)$.

Remark: **(C1)** is a condition on $\mathcal{D}$, not an assumption on (X, f, φ). Lusin's theorem $\Rightarrow$ suitable $\mathcal{D}$ exist for every ergodic μ.

Unbounded φ: same form of result, two extra conditions on $\mathcal{D}$

Specification

Topological transitivity $\Rightarrow$ for every $(x_1, n_1), \ldots, (x_k, n_k) \in X \times \mathbb{N}$ there exist $t_i \in \mathbb{N}$ and $x \in X$ such that for each $1 \leq j \leq k$,

$$f^{\sum_{i=0}^{j-1} n_i + t_i}(x) \in B_{n_j}(x_j, \varepsilon).$$

Definition

X has specification if for every $\varepsilon > 0$ there exists $\tau \in \mathbb{N}$ such that the above holds with $t_i \leq \tau$.

Key idea: if obstructions to specification have small pressure, they are invisible to equilibrium states

Non-uniform specification

Definition

$\mathcal{G} \subset X \times \mathbb{N}$ has specification at scale ε if there exists $\tau \in \mathbb{N}$ s.t. for every $(x_1, n_1), \dots, (x_k, n_k) \in \mathcal{G}$ there exist $t_i \le \tau$ and $x \in X$ such that $f^{\sum_{i=0}^{j-1} n_i + t_i}(x) \in B_{n_j}(x_j, \varepsilon)$ for each $1 \le j \le k$.

$\mathcal{G} \rightsquigarrow \mathcal{G}^M := \{(x, n) \mid (f^j(x), k) \in \mathcal{G}, 0 \le j, k \le M\}$
$\rightsquigarrow$ filtration $X \times \mathbb{N} = \bigcup_M \mathcal{G}^M$

Definition

$(\mathcal{P}, \mathcal{G}, \mathcal{S}) \subset (X \times \mathbb{N})^3$ is a decomposition for (X, f) if $\forall (x, n) \in X \times \mathbb{N}\ \exists p, g, s \in \mathbb{N}$ such that $p + g + s = n$ and

$$(x, p) \in \mathcal{P} \qquad (f^p x, g) \in \mathcal{G} \qquad (f^{p+g} x, s) \in \mathcal{S}$$

Choose decomposition such that every $\mathcal{G}^M$ has specification.

Uniqueness in the presence of small obstructions

Definition

The entropy of obstructions to specification at scale ε is

$$h^{\perp}_{\text{spec}}(\varepsilon) = \inf\{h(\mathcal{P} \cup \mathcal{S}, 3\varepsilon) \mid \exists \text{ decomposition } (\mathcal{P}, \mathcal{G}, \mathcal{S}) \text{ s.t. every } \mathcal{G}^M \text{ has specification at scale } \varepsilon\}$$

Theorem (C.–Thompson)

Let X be a compact metric space and $f : X \to X$ a continuous map. If there exists $\varepsilon > 0$ such that $h^{\perp}_{\text{exp}}(28\varepsilon) < h_{\text{top}}(f)$ and $h^{\perp}_{\text{spec}}(\varepsilon) < h_{\text{top}}(f)$, then there is a unique MME.

Extending this result requires an interpretation of pressure as a growth rate.

Classical variational principle

$$\left.\begin{array}{r} X \text{ compact} \\ f, \varphi \text{ continuous} \end{array}\right\} \Rightarrow \quad P^*(\varphi) = \sup_{\mathcal{D} \text{ top. sep.}} \left(\overline{\lim_{n\to\infty}} \frac{1}{n} \log \Lambda_n(\mathcal{D}, \varphi) \right)$$

All three conditions required

- X noncompact: may not be any invariant measures
- f discts: let $X = \{0,1\}^{\mathbb{N}}$, $Z = \{x \mid \text{limiting freq. of 1s DNE}\}$, $f = \sigma$ on Z and $f(x) = 0$ for $x \notin Z$. Then $h_{\text{top}}(f) = \log 2$ but only invariant measure is δ_0
- φ discts: X as above, $f = \sigma$, $\varphi = \mathbf{1}_Z$, then growth rate is $1 + \log 2$ but supremum of free energies is $\log 2$.

Proof of variational principle

Two halves of proof:

- **growth rate $\geq$ metric free energies** $P \geq P^*$
  If $p < h_\mu(f) + \int \varphi \, d\mu$, then there exists a topologically separated $\mathcal{D} \subset X \times \mathbb{N}$ such that $P(\mathcal{D}, \varphi) \geq p$.
- **metric free energies $\geq$ growth rate** $P^* \geq P$
  If $\mathcal{D}$ is topologically separated, then there exists $\mu \in \mathcal{M}_f(X)$ with $h_\mu(f) + \int \varphi \, d\mu \geq P(\mathcal{D}, \varphi)$

Note that counterexamples to naive generalisation all have $P^* < P$, not the other way around.

Claim: $P^*(\varphi) \leq P(\varphi)$ even without any assumptions on compactness or continuity.

Proving $P \geq P^*$

(1) Fix ergodic $\mu \in \mathcal{M}_f^{\varphi}(X)$

(2) $p < h_\mu(f) + \int \varphi \, d\mu \quad \longrightarrow \quad p = h + s \qquad \begin{cases} h < h_\mu(f) \\ s < \int \varphi \, d\mu \end{cases}$

(3) Use Birkhoff to get $\mathcal{C} \subset X \times \mathbb{N}$ such that for all large n

- $\mu(\mathcal{C}_n) \geq 1/2$
- $\frac{1}{n} S_n \varphi(x) \geq s$ for all $x \in \mathcal{C}_n$

(4) Use Katok entropy formula to show that if $\mu(\mathcal{C}_n) \geq 1/2$ for all large n, then for all $h < h_\mu(f)$ there is a topologically separated subset $\mathcal{D} \subset \mathcal{C}$ such that $\#\mathcal{D}_n \geq e^{nh}$ for large n

(5) Conclude $\Lambda_n(\mathcal{D}, \varphi) \geq (\#\mathcal{D}_n) e^{sn}$ and hence $P(\mathcal{D}, \varphi) \geq h + s$.

Proving $P^* \geq P(\mathcal{D}, \varphi)$

(1) $\mu_n = \sum_{x \in \mathcal{D}_n} a_{x,n} \mathcal{E}_{x,n}$ coefficients given by $a_{x,n} = \frac{e^{S_n \varphi(x)}}{\Lambda_n(\mathcal{D},\varphi)}$

(2) Weak*-convergent subsequence $\mu_{n_j} \to \mu$

(3) Observe that $\int \varphi \, d\mu_{n_j} \to \int \varphi \, d\mu$

(4) Check that μ is f-invariant

(5) Show that $h_\mu(f) + \int \varphi \, d\mu \geq P(\mathcal{D}, \varphi)$

1. Take partition $\alpha < \mathcal{U}$ with $\mu(\partial\alpha) = 0$
   $\Rightarrow \mu(\partial\alpha^q) = 0$ for all q $\Rightarrow$ $\mu_{n_j}(A) \to \mu(A)$ for all $A \in \alpha^q$
2. This + (top. sep. of $\mathcal{D}$) $\Rightarrow$ estimate $H_\mu(\alpha^q)$ and hence $h_\mu(f)$

Use of compactness and continuity in $P^* \geq P(\mathcal{D}, \varphi)$

(1) $\mu_n = \sum_{x \in \mathcal{D}_n} a_{x,n} \mathcal{E}_{x,n}$ coefficients given by $a_{x,n} = \frac{e^{S_n \varphi(x)}}{\Lambda_n(\mathcal{D},\varphi)}$

(2) Weak*-convergent subsequence $\mu_{n_j} \to \mu$ (X compact)

(3) Observe that $\int \varphi \, d\mu_{n_j} \to \int \varphi \, d\mu$ (φ continuous)

(4) Check that μ is f-invariant (f continuous)

(5) Show that $h_\mu(f) + \int \varphi \, d\mu \geq P(\mathcal{D}, \varphi)$

1. Take partition $\alpha < \mathcal{U}$ with $\mu(\partial\alpha) = 0$ (f continuous)
   $\Rightarrow \mu(\partial\alpha^q) = 0$ for all q $\Rightarrow$ $\mu_{n_j}(A) \to \mu(A)$ for all $A \in \alpha^q$
2. This + (top. sep. of $\mathcal{D}$) $\Rightarrow$ estimate $H_\mu(\alpha^q)$ and hence $h_\mu(f)$

Condition (C1)

Recall $P(\varphi) = \sup\{P(\mathcal{D}, \varphi) \mid \mathcal{D} \text{ top. sep. satisfying } \textbf{(C1)}\}$

(C1) Uniform tightness: $\forall \epsilon > 0$ there is compact $Z_\epsilon \subset X$ s.t.

- $\mathcal{E}_{x,n}(X \setminus Z_\epsilon) < \epsilon$ for all large n and $x \in \mathcal{D}_n$
- $f|_{Z_\epsilon}$ and $\varphi|_{Z_\epsilon}$ are continuous

Given $\mu \in \mathcal{M}_f^e(X)$, can strengthen step (3) from proof of $P \geq P^*$:

(3*) Lusin $\Rightarrow$ compact Z_ϵ s.t. $f|_{Z_\epsilon}, \varphi|_{Z_\epsilon}$ cts, $\mu(X \setminus Z_\epsilon) < \epsilon/2$.

$$\text{Birkhoff gives } \mathcal{C} \text{ s.t. } \begin{cases} \mu(\mathcal{C}_n) \geq 1/2 \\ \frac{1}{n} S_n\varphi(x) \geq s \text{ for all } x \in \mathcal{C}_n \\ \mathcal{E}_{x,n}(X \setminus Z_\epsilon) < \epsilon \text{ for all } x \in \mathcal{C}_n \end{cases}$$

Conclude that $P(\varphi) \geq P^*(\varphi)$: lose nothing by assuming **(C1)**

Utility of (C1) for μ_n and φ

$Z = \bigcup_{\epsilon>0} Z_\epsilon$

Define $Y \subset Z \times \mathbb{N}$ by

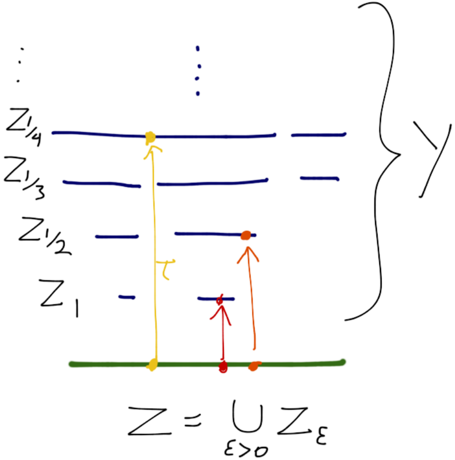
$Y = \bigcup_{\ell\in\mathbb{N}} Z_{1/\ell} \times \{\ell\}$

Lifting map $\tau : Z \to Y$

$\hat{\mu} = \mu \circ \tau^{-1}$

Projection $\pi : Y \to Z$

$\hat{\varphi} = \varphi \circ \pi$



Use of condition **(C1)** to prove $P^* \geq P(\mathcal{D}, \varphi)$

$Y = \bigcup_{\ell \in \mathbb{N}} Z_{1/\ell} \times \{\ell\} \subset Z \times \mathbb{N}$ $\quad \tau : Z \to Y$ $\quad \pi : Y \to Z$

(2) Convergent subseq. $\mu_{n_j} \to \mu$ $\quad$ ($\hat{\mu}_n = \mu_n \circ \tau^{-1} \in \mathcal{M}(Y)$ tight)

(3) $\int \varphi \, d\mu_{n_j} \to \int \varphi \, d\mu$ $\quad$ ($\hat{\varphi} = \varphi \circ \pi$ continuous on Y)

(4) Check that μ is f-invariant $\quad$ (f continuous on Z_ϵ)

(5) Show that $h_\mu(f) + \int \varphi \, d\mu \geq P(\mathcal{D}, \varphi)$

1. Take partition $\alpha < \mathcal{U}$ with $\mu(\partial\alpha) = 0$ $\quad$ (f continuous on Z_ϵ)
   $\Rightarrow \quad \mu_{n_j}(A) \to \mu(A)$ for all $A \in \alpha^q$
2. This + top. sep. of $\mathcal{D}$ $\Rightarrow$ estimate $H_\mu(\alpha^q)$ and hence $h_\mu(f)$

Invariance and entropy of μ

How to show $\mu = \lim_j \mu_{n_j}$ is f-inv.? Wk* top. metrisable,

$$D(\mu, f_*\mu) \leq D(\mu, \mu_n) + D(\mu_n, f_*\mu_n) + D(f_*\mu_n, f_*\mu)$$

- First term $\to 0$ since $\mu_{n_j} \to \mu$; second term $\leq 2/n$
- Third term $\to 0$ if f continuous

$(\mu(Z_\epsilon) > 1 - \epsilon)$ and $(f|_{Z_\epsilon}$ continuous$) \Rightarrow$ (third term $\to 0$)

(I)

Remains to estimate entropy $h_\mu(f)$

- $(\alpha < \mathcal{U}) + (\mathcal{D}$ sep. by $\mathcal{U}) \Rightarrow (\# A_w \cap \mathcal{D}_n \leq 1\ \forall w \in I^n)$
- Can choose $\alpha < \mathcal{U}$ with $\mu(\partial\alpha) = 0$

If f continuous, then $\mu(\partial\alpha^q) = 0$ hence $\mu_{n_j}(A_w) \to \mu(A_w)$

- Gives estimate on $H_\mu(\alpha^q)$, hence on $h_\mu(f)$

Using sets Z_ϵ can show that $\mu_{n_j}(A_w) \to \mu(A_w)$ given **(C1)**

(II)

Unbounded potential functions

$\varphi\colon X \to [-\infty,\infty]$: must guarantee $\int \varphi\, d\mu \geq \overline{\lim} \int \varphi\, d\mu_{n_j}$

Write $\varphi_{\geq K} = \varphi \mathbf{1}_{[\varphi \geq K]}$ and $\varphi_+ = \varphi_{\geq 0}$. Similarly $\varphi_{\leq K}$ and φ_-

(C2) $\frac{1}{n} S_n \varphi_-(x) \geq L > -\infty$ for all $(x,n) \in \mathcal{D}$ with n large

(C3) φ_+ is uniformly integrable: for all $\epsilon > 0$ there is $K > 0$ such that $\frac{1}{n} S_n \varphi_{\geq K}(x) < \epsilon$ for all $(x,n) \in \mathcal{D}$ with n large

Theorem

Let X be a complete separable metric space, $f: X \to X$ measurable, and $\varphi\colon X \to [-\infty,\infty]$ measurable. Let

$$P(\varphi) = \sup\{P(\mathcal{D},\varphi) \mid \mathcal{D} \text{ top. sep. and satisfies } \textbf{(C1)–(C3)}\}.$$

Then $P(\varphi) = \sup\{h_\mu(f) + \int \varphi\, d\mu \mid \mu \in \mathcal{M}_f(X), \varphi \in L^1(\mu)\}.$

Supremum over $\int \varphi \, d\mu > -\infty$

Previous result considers φ_- and φ_+ both integrable.

Alternative definition: only require φ_- to be integrable.

Require $\varphi^{-1}(+\infty)$ to be closed and allow broader class of $\mathcal{D}$:

(C3′) Either φ_+ is uniformly integrable (as before), or $\underline{\lim}_n \inf_{x \in \mathcal{D}_n} \mathcal{E}_{x,n}(\varphi^{-1}(+\infty)) > 0$, or $\mathcal{E}_{x,n}(\varphi_+)$ diverges uniformly ($\forall R > 0 \exists K > 0$ s.t. $\frac{1}{n} S_n \varphi_{\leq K}(x) \geq R \forall (x, n) \in \mathcal{D}$ with n large)

Theorem

Consider X a complete separable met. sp., $f : X \to X$ measurable, $\varphi \colon X \to [-\infty, \infty]$ measurable, $\varphi^{-1}(+\infty)$ closed. Let

$$P(\varphi) = \sup\{P(\mathcal{D}, \varphi) \mid \mathcal{D} \text{ top. sep. and satisfies } \textbf{(C1)},\textbf{(C2)},\textbf{(C3}'\textbf{)}\}.$$

Then $P(\varphi) = \sup\{h_\mu(f) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_f(X), \int \varphi \, d\mu > -\infty\}$.

Further questions

Gurevich pressure is equal to supremum over compact subshifts. Can one get an analogous result here?

- Probably require $P(\varphi) = \sup\{P(\mathcal{D}, \varphi) \mid \mathcal{D} \text{ has specification}\}$
- May also need extra regularity of φ (beyond continuity)

In compact case with bounded φ, let Y be the set of discontinuities for f and φ. Does $P(Y, \varphi) < P(X, \varphi)$ imply that supremum can be taken over all topologically separated $\mathcal{D}$ (ignoring **(C1)**)?

In classical setting, a non-uniform specification property gives uniqueness of equilibrium state as long as $P(\mathcal{C}, \varphi) < P(\varphi)$ for a certain $\mathcal{C} \subset X \times \mathbb{N}$. Does this go through for the more general notion of pressure as a growth rate?