A NOTE ON TWO APPROACHES TO THE THERMODYNAMIC FORMALISM

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Abstract. Inducing schemes provide a means of using symbolic dynamics to study equilibrium states of non-uniformly hyperbolic maps, but necessitate a solution to the liftability problem. One approach, due to Pesin and Senti, places conditions on the induced potential under which a unique equilibrium state exists among liftable measures, and then solves the liftability problem separately. Another approach, due to Bruin and Todd, places conditions on the original potential under which both problems may be solved simultaneously. These conditions include a bounded range condition, first introduced by Hofbauer and Keller. We compare these two sets of conditions and show that for many inducing schemes of interest, the conditions from the second approach are strictly stronger than the conditions from the first. We also show that the bounded range condition can be used to obtain Pesin and Senti's conditions for any inducing scheme with sufficiently slow growth of basic elements.

1. Introduction

Given a compact metric space X, a continuous map $f\colon X \to X$, and a continuous potential function $\varphi\colon X \to \mathbb{R}$, an *equilibrium measure* is a measure for which the supremum

$$P(\varphi) := \sup_{\mu\in\mathcal{M}} \left\{ h_\mu(f) + \int_X \varphi\, d\mu \right\}$$

is attained, where $\mathcal{M}$ is the class of all f-invariant Borel probability measures on X. The classical variational principle, due to Walters [Wal75], states that this supremum is equal to the topological pressure $P_{\text{top}}(\varphi)$, which is defined (without reference to a measure) as the growth rate of a certain *partition function.*

Interest in equilibrium states dates back to the work of Sinai, Ruelle, and Bowen in the 1970's, where it was established that if f is uniformly hyperbolic and topologically transitive, then any Hölder continuous potential function has a unique equilibrium state, which exhibits good ergodic and statistical properties [Bow75].

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A key element in this now classical work was the construction of a finite *Markov partition*, which allows one to obtain a semiconjugacy with a symbolic system on a finite alphabet, and then apply results from the theory of subshifts of finite type. Thus the finite Markov partition acts as a bridge between the original system and symbolic dynamics.

Beyond the uniformly hyperbolic case, the situation is somewhat more delicate. When f is only nonuniformly hyperbolic—for example, when f is an interval map that has a critical point but still exhibits non-zero Lyapunov exponents Lebesgue a.e.—it is typically the case that no finite Markov partition can be found, and so a new approach to existence and uniqueness of equilibrium states is required. There are also examples—such as the Manneville–Pomeau map or the logistic map $f(x) = 4x(1-x)$—for which a finite Markov partition can be found, but certain potentials of interest, such as the geometric potential $-\log|f'|$, do not correspond to Hölder continuous potentials in symbolic space, and thus the classical approach does not go through.

Thus it turns out to be of interest to generalise the classical results in (at least) two directions. One generalisation is to consider maps which are not uniformly hyperbolic; another is to find conditions on the potential other than Hölder continuity which may still yield good results.

An important idea in both these directions is the *bounded range condition* (given below as **(BT1)**), which demands that the difference between the supremum and infimum of φ not be more than the topological entropy of the map. This condition was introduced by Hofbauer and Keller [HK82] in the context of piecewise monotonic maps of the interval (which may not be uniformly hyperbolic), where it was used to prove the existence (but not necessarily uniqueness) of an equilibrium state with good ergodic properties, even in the case when φ is not Hölder continuous. This condition has since been used by Denker and Urbański [DU91], Oliveira [Oli03], Varandas and Viana [VV08], and Bruin and Todd [BT08] in a variety of contexts. (See also [Hof77] and [Hu08] for a discussion of certain classes of non-Hölder potentials.)

On the symbolic side of things, the most important extension of the classical results involves considering symbolic systems on countably infinite alphabets; this theory of *countable Markov shifts* was developed by Buzzi and Sarig [BS03, Sar99, Sar03], by Mauldin and Urbański [MU01], and by Yuri [Yur99]. In this setting, Hölder continuity of the potential function φ is no longer enough to guarantee the existence of a unique equilibrium state, and new conditions on φ must be introduced and verified.

To begin with, the classical notion of topological pressure must be replaced by the *Gurevich pressure*. The classical pressure is the growth rate of a partition function which involves a sum over a maximal (n, ε)-separated set; for a symbolic space with a finite alphabet, it can be shown that one may also define the partition function via a sum over all periodic orbits of length n. To define the Gurevich pressure, one uses this method, but the sum is only taken over those orbits lying in a fixed 1-cylinder. If φ has *finite Gurevich pressure* and satisfies a further condition on the growth rate of a related partition function, known as *positive recurrence* (the precise definition will not be needed for our purposes), then we regain existence and uniqueness.

Similarly, if we consider non-symbolic systems without finite Markov partitions, the assumption of Hölder continuity of the potential $\varphi\colon X \to \mathbb{R}$ is no longer sufficient to guarantee existence and uniqueness, and further conditions on φ are required; in this context the bounded range condition mentioned above becomes important.

The bridge between the original system and symbolic dynamics is provided by the concept of *inducing schemes*, which generalise the classical notion of first return maps, and were considered by Schweiger in [Sch75, Sch81] under the name *jump transformations*, and by Kakutani, Rokhlin, and others (see [PS08] for further details). These involve restricting our attention to a subset $W \subset X$, called the *base* of the inducing scheme, and an *induced map*

$$\begin{aligned} F\colon\ W &\to W, \\ x &\mapsto f^{\tau(x)}(x), \end{aligned}$$

where $\tau(x)$ is the *inducing time*; $\tau(x)$ is a return time to W, but is generally *not* the first return time. With a suitable choice of W and τ, the induced map F can in many cases be made equivalent to the full shift on a countable alphabet, whose symbols correspond to certain subsets of W on which τ is constant.

Using the correspondence between the induced map F and the full shift on a countable alphabet, one obtains a correspondence between potentials on X and *induced potentials* on symbolic space, and between measures on X and *lifted measures* on symbolic space. This correspondence, however, is incomplete, and there are issues of *liftability* of measures; there may be measures on X which do not correspond to a measure on symbolic space. These issues are explained and addressed in [PSZ08], as well as in [PS08, BT08, BT09]—the results of the latter three papers are our primary concern here.

Working from the symbolic side of things, Pesin and Senti [PS08] give conditions on the *induced potential* $\overline{\varphi}$ which allow them to apply Sarig's results and show the existence of a unique equilibrium state. These conditions apply to a very large class of maps with inducing schemes, including multidimensional maps—however, as the approach only produces a unique equilibrium state *within the class of liftable measures*, the class of maps for which they obtain complete results, and show that there is a unique equilibrium state *among all invariant measures*, is much smaller. In particular, they build inducing schemes for certain maps of the interval, including in particular transverse one-parameter families of unimodal maps, with respect to which any equilibrium state must be liftable (for certain potentials).

Working from perspective of the original system, Bruin and Todd [BT08] use inducing schemes to study a class of one-dimensional maps which is more general than that considered by Pesin and Senti; they show that Sarig's results can also be applied to a certain class of potential functions φ which satisfy the bounded range condition. Because they address the liftability problem by considering many different inducing schemes, their conditions are formulated on the original potential φ, independently of any inducing scheme. For the inducing schemes they build, these conditions allow them to use Sarig's results; for other inducing schemes on the same class of maps, or for broader classes of maps, the bounded range condition **(BT1)** continues to be relevant (see below), while it is not as clear what role the second condition **(BT2)** plays.

The purpose of this brief note is to examine the relationship between the two sets of conditions just mentioned. This relationship may be summarised as follows (see Section 2 for the relevant definitions and precise statements):

(1) For the inducing schemes Bruin and Todd consider, their conditions imply Pesin and Senti's (the proof of this is implicit in [BT08], but is not formally stated).
(2) For *any* inducing scheme satisfying a certain liftability condition and a certain estimate on the growth rate of basic elements, the bounded range condition alone is enough to imply Pesin and Senti's conditions.
(3) For a very broad class of inducing schemes, one may construct a potential function which does not have bounded range, but whose induced potential satisfies Pesin and Senti's conditions.

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2. Definitions and statement of results

We denote the dynamical system by $f\colon I \to I$, and write W for the base of the inducing scheme.

Denote by $\mathcal{M}_L$ the class of measures which are *liftable* to the inducing scheme;[1] the first step in showing the existence of a unique equilibrium state for φ is to establish conditions under which the functional $h_\mu(f) + \mu(\varphi)$ can be shown to obtain a unique maximum *among measures in* $\mathcal{M}_L$, using results of Sarig [Sar99, Sar03]. Such a measure μ is known as a *liftable* equilibrium state; in order to establish that μ is the unique equilibrium state within the class of *all* invariant measures, one must build an inducing scheme with

[1] Bruin and Todd refer to such measures as *compatible*. The precise definition of liftability (compatibility) will not concern us here.

respect to which any equilibrium state is liftable. This is accomplished by rather different methods in [PS08] and in [BT08, BT09], but this has more to do with the construction of the inducing schemes themselves, rather than with the conditions on the potentials, and so the distinction will not be important for our purposes; rather, we compare the manner in which the two approaches guarantee that Sarig's results apply, giving the existence of a unique liftable equilibrium measure.

In order to give the precise statements of the conditions we wish to compare, we must introduce some more notation and terminology.

Let $\varphi\colon I \to \mathbb{R}$ be the potential function, which need not be continuous, and consider the following related functions:

(1) The *induced* potential $\overline{\varphi}\colon W \to \mathbb{R}$, defined by
$$\overline{\varphi}(x) := S_{\tau(x)}\varphi(x),$$
where $S_n\varphi(x)$ denotes the n^{th} ergodic sum $\sum_{k=0}^{n-1}\varphi(f^k(x))$.

(2) The *centred* potential $\psi := \varphi - P_L(\varphi)$, where
$$P_L(\varphi) := \sup_{\mu\in\mathcal{M}_L}\left\{h_\mu(f) + \int_X \varphi\, d\mu\right\}$$
is the restricted pressure obtained by taking the supremum over all *liftable* measures. *A priori*, $P_L(\varphi)$ may be infinite, but once it is shown to be finite, ψ can be defined, and has the property that $P_L(\psi) = 0$, which proves to be useful in applying Abramov's and Kac's formulae [PS08, Theorem 2.3]. In particular, [PS08, Theorem 4.2] shows that $P_L(\varphi) < \infty$ if $\overline{\varphi}$ satisfies (SV) below and has finite Gurevich pressure.

(3) If $P_L(\varphi) < \infty$, then one can also define the *induced centred* potential $\varphi^+ := \overline{\varphi - P_L(\varphi)}$.

Let S be the set of basic intervals for the inducing scheme; that is, $W = \bigcup_{J\in S} J$, where each $J \in S$ is an interval on which the inducing time τ is constant, and the different elements of S are disjoint (we will use the notation (S,τ) to identify a particular inducing scheme). Then the n^{th} variation of the induced potential with respect to the cylinders generated by the inducing scheme is
$$V_n(\overline{\varphi}) = \sup_{(J_1,\dots,J_n)\in S^n} \sup\{|\overline{\varphi}(x) - \overline{\varphi}(y)| \mid F^k(x), F^k(y) \in J_{k+1} \text{ for all } 0 \le k < n\},$$
and we say that the induced potential has *summable variations* if
$$\sum_{n=1}^{\infty} V_n(\overline{\varphi}) < \infty, \tag{SV}$$

The basic result in both sets of papers, which establishes the existence of a unique liftable equilibrium state, is the following, which is [PS08, Theorem 4.4].

Proposition 1. *Assume that the induced potential $\overline{\varphi}$ has summable variations* (SV)*, finite Gurevich pressure, and is positive recurrent. If the induced centred potential φ^+ satisfies*

$$\sup_{J\in S}\sup_{x\in\overline{J}}\varphi^+(x)<\infty,$$

then there exists an invariant ergodic Gibbs measure ν_{φ^+} on W for the induced map F. Furthermore, if the inducing time is integrable with respect to this measure—that is, if $\sum_{J\in S}\tau(J)\nu_{\varphi^+}(J)<\infty$—then this measure lifts to a measure μ on I which is the unique liftable equilibrium measure.

Both sets of papers use Proposition 1 in a fundamental way, although it is not formulated as a separate result in [BT08, BT09]. The papers diverge when it comes to establishing conditions on the potential function under which these hypotheses of Proposition 1 are fulfilled; it is these conditions in which we are interested at present.

The condition (SV) appears in unaltered form in both approaches, and so we will have nothing further to say about it here.[2]

Pesin and Senti place the following conditions on the induced and induced centred potentials:

(PS1): There exists $c\in\mathbb{R}$ such that

$$\sum_{J\in S}\sup_{x\in J}\exp((\overline{\varphi-c})(x))<\infty.$$

(PS2): There exists $\varepsilon_0>0$ such that

$$\sum_{J\in S}\tau(J)\sup_{x\in J}\exp(\varphi^+(x)+\varepsilon_0\tau(x))<\infty.$$

Before stating the consequences of these conditions, a few remarks on the conditions themselves are in order. First we observe that **(PS1)** is stated in [PS08] with the more restrictive requirement $c=0$. However, since $V_n(\overline{\varphi-c})=V_n(\overline{\varphi})$ and $(\varphi-c)^+=\varphi^+$, shifting the potential by a constant has no effect on (SV) or on **(PS2)**, and since φ and $\varphi-c$ have the same equilibrium states, there is no loss of generality here.

Secondly, we observe that **(PS1)** follows immediately from **(PS2)** by setting $c=P_L(\varphi)$, and so is in some sense redundant once all is said and done. However, since there is no *a priori* guarantee that $P_L(\varphi)$ is finite, one must deal with the possibility that **(PS2)** may not even make sense to state, since φ^+ is not defined if $P_L(\varphi)=\infty$.

If φ is bounded above, then $P_L(\varphi)\leq h_{\text{top}}(f)+\sup\varphi<\infty$, and so **(PS1)** may be dispensed with. Since most of the potentials we consider in this

[2]In fact, Pesin and Senti require the slightly stronger condition that $V_n(\overline{\varphi})$ decays exponentially, or equivalently, that $\overline{\varphi}$ is Hölder. However, their proof of existence and uniqueness for equilibrium states depends only on the fact that this implies (SV); the stronger requirement that the rate of decay be exponential only becomes important in the proof of certain statistical properties.

paper are bounded above as a consequence of the bounded range condition **(BT1)** introduced below, we will be primarily concerned with the condition **(PS2)**.

For unbounded potentials, such as the geometric potential $\varphi(x) = -\log|f'(x)|$ for an interval map with a critical point, one must first show that φ^+ is properly defined, which Pesin and Senti do as follows: **(PS1)** implies that $\overline{\varphi - c}$ has finite Gurevich pressure, and hence $P_L(\varphi - c) < \infty$ [PS08, Theorem 4.2], from which we have $P_L(\varphi) = P_L(\varphi - c) + c < \infty$ as well.

Pesin and Senti prove the following result [PS08, Theorem 4.5]:

Theorem 2. *Let f be a continuous map of a compact metric space, with finite topological entropy, and fix an inducing scheme for f.*[3] *Let φ be a potential function satisfying* (SV)*,* ***(PS1)****, and* ***(PS2)****. Then there exists a unique liftable equilibrium measure for φ.*

Pesin and Senti explicitly construct a particular inducing scheme, and then show that any equilibrium measure must be liftable to this inducing scheme; thus if Theorem 2 holds, then the unique liftable equilibrium measure with respect to this inducing scheme is in fact a true unique equilibrium measure.

Bruin and Todd, on the other hand, construct a broad class of inducing schemes [BT08, Proposition 1], which we shall denote $\mathcal{S}$, and then show that any measure with positive Lyapunov exponent is liftable to one of these. Since they consider more than one inducing scheme, they place the following conditions on the potential φ itself (rather than the induced potential):

(BT1): Bounded range,
$$\sup \varphi - \inf \varphi < h_{\text{top}}(f).$$

(BT2):
$$\tilde{V}_n(\varphi) \to 0,$$
where $\tilde{V}_n$ is the n^{th} variation with respect to the cylinders generated by the branch partition (the branch partition is the partition of I into maximal intervals of monotonicity for f). In particular, this implies that φ is continuous away from the preimages of the critical point.

They then prove the following [BT08, Theorem 4]:

Theorem 3. *Let f be a topologically mixing C^2 interval for which all periodic points are repelling and all critical points are non-flat. If a potential φ satisfies* ***(BT1)*** *and* ***(BT2)****, and if the induced potential for every inducing scheme in $\mathcal{S}$ satisfies* (SV)*, then*

(a) there exists a unique equilibrium state μ_φ for φ;

[3] Pesin and Senti place certain requirements on the inducing scheme, which do not concern us.

(b) μ_φ *is liftable to an inducing scheme* (S,τ) *for which the inducing time* τ *is such that the tails* $\nu_{\varphi^+}(\{x \mid \tau(x) \geq n\})$ *decrease exponentially in* n*, where* ν_{φ^+} *is the equilibrium state (on* W*) for the induced potential* φ^+*.*

Using the Gibbs property of ν_{φ^+}, we will prove the following in Section 3.

Theorem A. *Let* f *and* φ *be as in Theorem 3, and let* $(S,\tau) \in \mathcal{S}$ *be an inducing scheme with exponential tails, as in part (b) of Theorem 3. Then the induced potential* $\overline{\varphi}$ *satisfies* ***(PS1)*** *and* ***(PS2)****.*

For broader classes of inducing schemes, such as those built by Pesin and Senti, the condition **(BT2)** is irrelevant, as its utility comes from the fact that the inducing schemes in $\mathcal{S}$ each have a cylinder in the branch partition as their base. However, the bounded range condition **(BT1)** is useful in a broader setting. In particular, it turns out to imply Pesin and Senti's conditions for many inducing schemes, including those constructed in [PS08]. We need the following definition.

Definition 1. Consider an inducing scheme (S,τ); for each $n \in \mathbb{N}$, let $S(n)$ be the number of 1-cylinders J in the base of the inducing scheme for which $\tau(J) = n$. The *growth rate of basic elements* of the inducing scheme is

$$\gamma(S,\tau) = \overline{\lim}_{n\to\infty} \frac{1}{n} \log S(n).$$

Theorem B. *Consider a continuous map* $f\colon X \to X$ *of a compact metric space, and let* $\varphi\colon X \to \mathbb{R}$ *be a potential which satisfies the bounded range condition* ***(BT1)****. Let* (S,τ) *be any inducing scheme for which the following two conditions hold:*

(1) Some measure of maximal entropy is liftable.
(2) $\gamma(S,\tau) < h_{\mathrm{top}}(f) - (\sup\varphi - \inf\varphi)$*.*

Then the induced potential $\overline{\varphi}$ *satisfies* ***(PS1)*** *and* ***(PS2)****.*

We remark that the inducing schemes constructed by Pesin and Senti (for one-parameter families of unimodal maps) are built so that a measure of maximal entropy is liftable [PS08, Theorems 7.4 and 7.6]. Furthermore, given any $\eta > 0$, they find a set of parameters of positive Lebesgue measure such that for each parameter, an inducing scheme can be found with $\gamma(S,\tau) < \eta$. For such inducing schemes, Theorem B shows that the bounded range condition is sufficient to guarantee **(PS1)** and **(PS2)**.

We also remark that the conditions of the theorem may be weakened slightly: the assumption that some measure of maximal entropy is liftable may be weakened to the assumption that there exists a liftable measure $\mu \in \mathcal{M}_L$ such that $h_\mu(f) > \sup\varphi - \inf\varphi + \gamma(S,\tau)$. This will be evident from the proof.

Finally, we observe that for a very large class of inducing schemes, there exists a potential which fails to satisfy **(BT1)**, but which has an induced potential satisfying **(PS1)** and **(PS2)**.

Given a continuous map $f\colon X \to X$ of a compact metric space, denote by $\mathcal{A}$ the class of inducing schemes (S, τ) such that

(1) some measure of positive entropy is liftable to S;
(2) $S(n)$ is finite for every n;
(3) There exists n such that if x and y lie in the same n-cylinder of the induced map (that is, $F^k(x)$ and $F^k(y)$ lie in the same element $J \in S$ for all $0 \leq k < n$), then x and y have the same pattern of returns to W prior to their inducing time $\tau(x) = \tau(y)$; that is, if $1 \leq i < \tau(x)$ and $f^i(x) \in J \in S$, then $f^i(y) \in J$ as well.

Note that the first two requirements are very weak. We do not require liftability of a measure of maximal entropy, or of *every* measure of positive entropy; liftability of a single measure with positive entropy suffices. Similarly, we do not require any uniform bound on $S(n)$; we may very well have $S(n) \to \infty$ (arbitrarily quickly). All that is required is finiteness of the number of basic elements $S(n)$ with each particular inducing time n.

The third requirement is automatically satisfied if τ is the first return time to W; this is the case for the inducing schemes built to study the Manneville–Pomeau map, for example.

Theorem C. *For every inducing scheme $(S, \tau) \in \mathcal{A}$, there exists a potential $\varphi\colon X \to \mathbb{R}$ such that $\overline{\varphi}$ satisfies* (SV)*,* ***(PS1)****, and* ***(PS2)****, but φ does not satisfy* ***(BT1)****.*

3. Proofs

Proof of Theorem A. Since φ is bounded above, we immediately have $P_L(\varphi) \leq h_{\text{top}}(f) + \sup \varphi < \infty$, and hence φ^+ is properly defined.

To show that **(PS2)** holds, we use the result of Theorem 3(b), which guarantees the existence of a Gibbs measure ν_{φ^+} and constants $\gamma, C_1 > 0$ such that

$$\nu_{\varphi^+}(\{x \in W \mid \tau(x) \geq n\}) \leq C_1 e^{-\gamma n}$$

for all $n \geq 1$. Because ν_{φ^+} is a Gibbs measure for φ^+, there exists $C_2 > 0$ such that for every $x \in J \in S$, we have

$$\frac{1}{C_2} \leq \frac{\nu_{\varphi^+}(J)}{\exp(P_G(\varphi^+) + \varphi^+(x))} \leq C_2.$$

Thus $\exp(\varphi^+(x)) \leq C_3 \nu_{\varphi^+}(J)$, where $C_3 = C_2 e^{-P_G(\varphi^+)}$; it follows that

$$\sum_{\substack{J \in S \\ \tau(J) \geq n}} \sup_{x \in J} \exp(\varphi^+(x)) \leq C_4 e^{-\gamma n},$$

for every n, where $C_4 = C_3 C_1$. In particular, we have

$$\sum_{\substack{J \in S \\ \tau(J) = n}} \sup_{x \in J} \exp(\varphi^+(x)) \leq C_4 e^{-\gamma n},$$

and thus given $0 < \varepsilon < \gamma$,

$$\sum_{\substack{J\in S\\ \tau(J)=n}} n \sup_{x\in J} \exp(\varphi^+(x) + \varepsilon n) \le C_4 n e^{-(\gamma-\varepsilon)n}.$$

Summing over n, we obtain

$$\begin{aligned}\sum_{J\in S} \tau(J) \sup_{x\in J} \exp(\varphi^+(x) + \varepsilon\tau(x)) \\ &\le \sum_{n\ge 1} \sum_{\substack{J\in S\\ \tau(J)=n}} n \sup_{x\in J} \exp(\varphi^+(x) + \varepsilon n) \\ &\le \sum_{n\ge 1} C_4 n e^{-(\gamma-\varepsilon)n} < \infty,\end{aligned}$$

and so **(PS2)** holds. (**(PS1)** then follows by setting $c = P_L(\varphi)$.) □

Proof of Theorem B. As in the proof of Theorem A, the bounded range condition immediately implies $P_L(\varphi) < \infty$, so φ^+ is properly defined. From the assumption that a measure of maximal entropy is liftable and from the assumption on the growth rate of $S(n)$, we see from **(BT1)** that

$$\begin{aligned}P_L(\varphi) &= \sup_{\mu\in\mathcal{M}_L} \{h_\mu(f) + \mu(\varphi)\} \ge \left(\sup_{\mu\in\mathcal{M}_L} h_\mu(f)\right) + \inf\varphi \\ &= h_{\text{top}}(f) + \inf\varphi > \sup\varphi + \gamma(S,\tau).\end{aligned}$$

It follows that

$$\varphi^+(x) = S_{\tau(x)}\varphi(x) - P_L(\varphi)\tau(x) \le (\sup\varphi - P_L(\varphi))\tau(x) = -\alpha\tau(x),$$

where $\alpha := P_L(\varphi) - \sup\varphi > \gamma(S)$. Fixing $\eta \in (\gamma(S,\tau), \alpha)$, we observe that by the definition of $\gamma(S)$, there exists $N \in \mathbb{N}$ such that $\log S(n) \le n\eta$ for all $n \ge N$. It follows that for $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{J\in S} \tau(J) \sup_{x\in J} \exp(\varphi^+(x) + \varepsilon\tau(x)) &= \sum_{n=1}^{\infty} \sum_{\substack{J\in S\\ \tau(J)=n}} n \sup_{x\in J} \exp(\varphi^+(x) + \varepsilon n) \\ &\le \sum_{n=1}^{\infty} nS(n) \exp((-\alpha+\varepsilon)n) \\ &\le \sum_{n=1}^{N-1} nS(n) \exp((-\alpha+\varepsilon)n) + \sum_{n=N}^{\infty} n \exp((-\alpha+\varepsilon+\eta)n),\end{aligned}$$

and since $\eta < \alpha$, we may find $\varepsilon > 0$ such that $-\alpha + \varepsilon + \eta < 0$. Thus **(PS2)** holds, and **(PS1)** follows. □

Proof of Theorem C. Consider the function $r\colon \mathbb{N} \to \mathbb{R}^+$ given by $r(k) = k(k + \log S(k))$, and define a potential function $\varphi\colon X \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} -r(\tau(x)) & x \in W, \\ -1 & x \notin W. \end{cases}$$

(Recall that $W = \bigcup_{J\in S} J$ is the base of the inducing scheme (S, τ).)

Now consider the one-parameter family of potential functions $t\varphi$. Observe that for every invariant measure μ, we have $\int t\varphi \, d\mu \le -t$, and hence

$$\begin{aligned} P_L(t\varphi) &= \sup_{\mu\in\mathcal{M}_L} \left\{ h_\mu(f) + \int_X t\varphi \, d\mu \right\} \\ &\le \left(\sup_{\mu\in\mathcal{M}_L} h_\mu(f) \right) - t \\ &\le h_{\text{top}}(f) - t. \end{aligned}$$

Thus $P_L(t\varphi) \le 0$ for all $t \ge h_{\text{top}}(f)$. Furthermore, since some measure of positive entropy is liftable, we have $0 < P_L(0) < \infty$, and it follows from continuity of the map $t \mapsto P_L(t\varphi)$ that there exists $t \in (0, h_{\text{top}}(f)]$ such that $P_L(t\varphi) = 0$. Thus $(t\varphi)^+ = \overline{t\varphi}$.

The induced potential for $t\varphi$ is

$$\overline{t\varphi}(x) = t(\varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{\tau(x)-1}(x))).$$

For $1 \le i \le \tau(x) - 1$, we have $\varphi(f^i(x)) \le -1$; hence writing $k = \tau(x)$, we have

$$\overline{t\varphi}(x) \le -t(r(k) + k - 1). \tag{1}$$

Now we can verify the conditions on $\overline{t\varphi}$. First we observe that by the third condition on inducing schemes in $\mathcal{A}$, we have $V_k(\overline{t\varphi}) = 0$ for all $k \ge n$, and it follows that $\overline{t\varphi}$ satisfies (SV).

To show **(PS1)** and **(PS2)**, observe that by (1), we have

$$\begin{aligned} \sum_{J\in S} \sup_{x\in J} \exp(\overline{t\varphi}(x)) &\le \sum_{k\ge1} S(k) e^{-t(r(k)+k-1)}, \\ \sum_{J\in S} \tau(J) \sup_{x\in J} \exp((t\varphi)^+(x) + \varepsilon\tau(x)) &\le \sum_{k\ge1} kS(k) e^{-t(r(k)+k-1)+\varepsilon k}. \end{aligned} \tag{2}$$

From our choice of r, we have

$$\frac{r(k)}{k + \log S(k)} = k,$$

and so

$$\frac{r(k)}{k + \log S(k)} > \frac{1}{t}$$

for all sufficiently large k. It follows that

$$tr(k) > k + \log S(k),$$

whence

$$S(k)e^{-tr(k)} < e^{-k},$$

which implies that the sums in (2) both converge, and so **(PS1)** and **(PS2)** hold.

Now if τ is unbounded, then $t\varphi$ is unbounded, and hence **(BT1)** fails, which completes the proof in this case.

If τ is bounded, then $t\varphi$ satisfies **(BT1)** for small values of t. However, by the assumption that $S(n)$ is finite for each n, we see that (S, τ) only has finitely many basic elements, and so the sums in **(PS1)** and **(PS2)** are finite for any value of t. By choosing t sufficiently large, we can find $t\varphi$ such that **(BT1)** is not satisfied. $\square$

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