Randomness and determinism in dynamical systems

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The talk in one slide

PHENOMENON	Deterministic systems can exhibit stochastic behaviour over long time scales
KNOWN	Mechanism driving this is phase space expansion
EXAMPLES	Lorenz equations, expanding maps, logistic map
RESEARCH	What happens when expansion is non-uniform?

Predictions in dynamical systems

Key objects:

- X = phase space for a dynamical system.
  Points in X correspond to configurations of the system.
- $f: X \circlearrowleft$ describes evolution of the state of the system over a single time step. *Can also consider continuous-time systems.*

Standing assumptions:

- $X \subset \mathbb{R}^n$
- f is continuous

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Predictions rely on finding $f^n(x)$ given x.

initial error $\Rightarrow$ must compare $f^n(x)$ and $f^n(y)$ when $x \sim y$

Distinct problem from accounting for discrepancy between model and real-world system, or for numerical error in computation of $f^n(x)$.

A mechanism for stochastic behaviour

Fix $x \sim y$. Two extremes:

- Stable behaviour: $d(f^n x, f^n y) \to 0$
  Even better: there is $p = f(p)$ such that $f^n x \to p$ for all x
- Unstable behaviour: $d(f^n x, f^n y)$ grows quickly

In "chaotic" systems, unstable behaviour is prevalent:

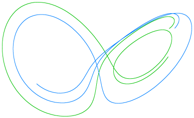
- initial error grows exponentially fast
- prediction $f^n(x)$ quickly diverges from reality

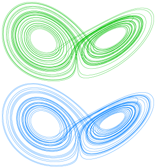
Another perspective: $U \subset X$ a small neighbourhood, consider $f^n(U)$.

- In chaotic systems, diameter of iterates $f^n(U)$ becomes large (exponentially quickly) no matter how small U is.

Lorenz equations (1963) – atmospheric dynamics

$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$
$$\dot{y} = x(\rho - z) - y \qquad \rho = 28$$
$$\dot{z} = xy - \beta z \qquad \beta = 8/3$$





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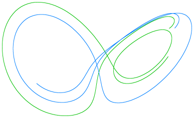

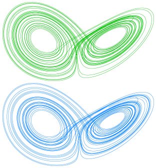

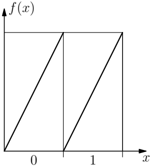

Doubling map $f\colon S^1 \circlearrowleft$, $S^1 \subset \mathbb{C}$, $z = e^{ix} \mapsto z^2 = e^{i(2x)}$

Full shift $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, $f = \sigma\colon x_0x_1x_2\ldots \mapsto x_1x_2x_3\ldots$

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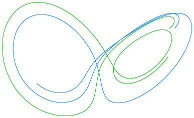

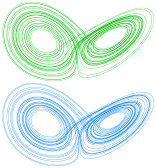

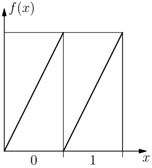

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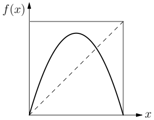
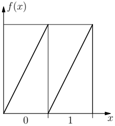
Logistic map $f_\lambda\colon [0,1] \circlearrowleft, x \mapsto \lambda x(1-x),\ \lambda \in [0,4]$

Code trajectories with 0s and 1s, but don't get full shift.



Predictions for the doubling map



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Predictions are impossible: If initial error is ϵ then error at time n is $\epsilon 2^n$.

- Lengthening prediction by time 1 requires doubling initial accuracy.

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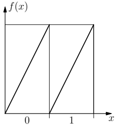

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Predictions are easy: Lebesgue measure ν on the circle is f-invariant

$$\nu(f^{-1}E) = \nu\{z \mid f(z) \in E\} = \nu(E) \text{ for every measurable } E \subset S^1$$

It is also ergodic: if $f^{-1}(E) = E$ then $\nu(E) = 0$ or 1.

Birkhoff ergodic theorem: for every $\varphi \in L^1(S^1)$ and ν-a.e. $z \in S^1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k z) = \int_{S^1} \varphi(y)\, d\nu(y)$$

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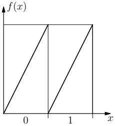

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LAW OF LARGE NUMBERS

A Bernoulli process

Lebesgue measure on the circle passes to a measure μ on $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$

- $\mu(E) = \nu(\pi(E))$, *where* $\pi\colon \Sigma_2^+ \to S^1,\ x \mapsto \exp(\pi i \sum_{k=0}^{\infty} x_k 2^{-k})$

Define $\varphi\colon \Sigma_2^+ \to \mathbb{R}$ by $\varphi(x) = x_0$. This gives a sequence of random variables on (Σ_2^+, μ) by $X_n = \varphi(f^n x)$.

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- Central limit theorem: $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_k - \mathbb{E}X)$ converges to Gaussian
- Large deviations: Estimates on $P(|\frac{1}{n} \sum_{k=1}^{n} X_k - \mathbb{E}X| > \delta)$
- Law of the iterated logarithm: $\frac{\sum_{k=1}^{n}(X_k - \mathbb{E}X)}{\sqrt{n \log\log n}}$ converges to zero in probability but not almost surely
- . . . and so on . . .

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This works for any “nice enough” φ, and all this happens **even though the dynamical system is deterministic.**

Abundance of invariant measures

IDEA — Deal with chaotic behaviour by treating the observations $\varphi \circ f^k$ as random variables.

Requires an invariant measure μ, and Σ_2^+ has many such measures.

- Bernoulli measures – weighted coin flips ν_r, $r \in (0,1)$
- Periodic orbit measures – atomic $\delta_{\mathcal{O}(p)} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k p}$
- Everything in between: Markov measures, Gibbs measures, etc.

Some have good statistical properties, some don't. **Which are natural?**

Look for an absolutely continuous invariant measure (acim).

Expansion and contraction

Can we deal with the logistic map $f_\lambda(x) = \lambda x(1-x)$ this way?

- Find acim μ, treat $X_n = \varphi(f^k x)$ as a stochastic process.

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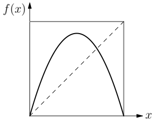
Doubling map has uniform expansion: $d(fx, fy) = 2d(x, y)$ if $x \sim y$

- Destroys correlations and yields stochastic behaviour

Logistic map has both expansion and contraction:

- $d(fx, fy) < d(x, y)$ if x, y near critical point
- $d(fx, fy) > d(x, y)$ if away from critical point



How much time does a typical orbit spend near critical point?

Typical orbits for logistic map

Consider logistic map $f(x) = 4x(1-x)$. "Typical" means w.r.t. Lebesgue, but now Lebesgue measure is not invariant.

- Fact: $d\mu = \pi^{-1}(x(1-x))^{-1/2}\,dx$ is an ergodic invariant measure
- Claim: $\exists\chi > 0$ such that typical points x have $|(f^n)'(x)| \approx e^{\chi n}$

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$$
\begin{aligned}
\frac{1}{n}\log|(f^n)'(x)| &= \frac{1}{n}\sum_{k=0}^{n-1}\log|f'(f^k x)| && \text{(chain rule)} \\
&\xrightarrow{\text{Leb-a.e.}} \int_0^1 \log|f'(y)|\, d\mu(y) && \text{(ergodic theorem)} \\
&= \frac{1}{\pi}\int_0^1 \frac{\log|4-8y|}{\sqrt{y(1-y)}}\, dy && \text{(definition of } f,\mu\text{)} \\
&= \log 2 && \text{(wizardry)}
\end{aligned}
$$

Dependence on parameter value

The parameter λ in $f_\lambda(x) = \lambda x(1-x)$ ranges from 0 to 4.

- When $\lambda = 4$, expansion beats contraction for typical orbits.
- For $0 \leq \lambda \leq 3$, there is an attracting fixed point (contraction wins). No stochastic behaviour in this case.

$$x = f_\lambda(x) = \lambda x(1-x) \quad \Leftrightarrow \quad x = 0, 1 - \frac{1}{\lambda}$$

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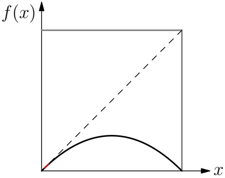

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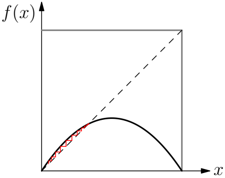

$\lambda = 1.5$

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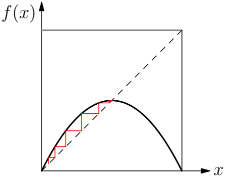

$\lambda = 2$

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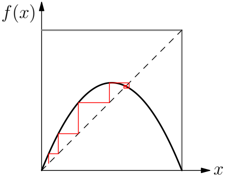

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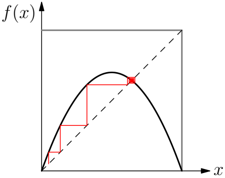

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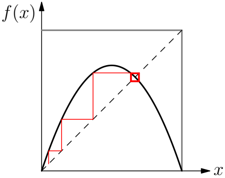

$\lambda = 3$

What happens for $3 < \lambda < 4$? Which is dominant, expansion or contraction?

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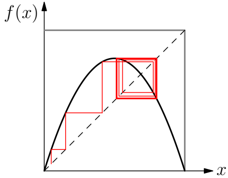

$\lambda = 3.2$

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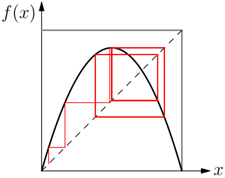

$\lambda = 3.5$

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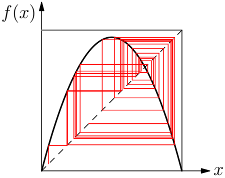

$\lambda = 3.8$

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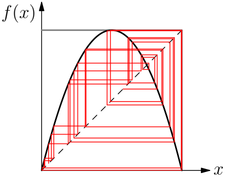

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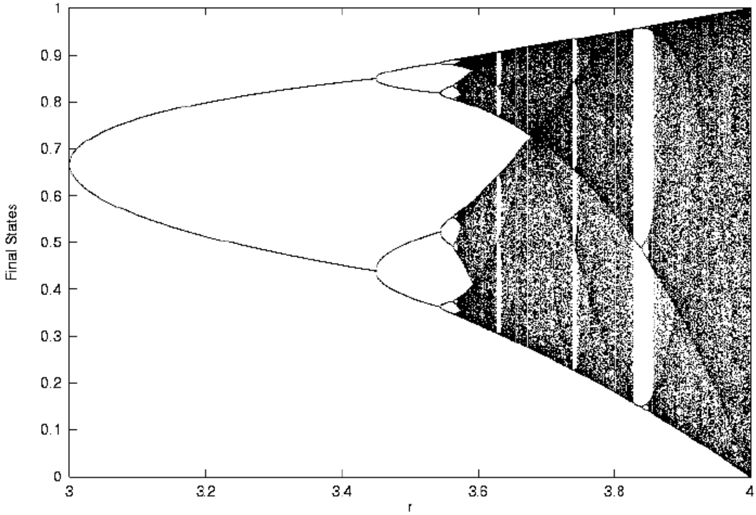
Final States
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3.2
3.4
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3.8
4
r

Classification of behaviour

At least two types of behaviour:

1. Attracting periodic orbit: *$f^p(x) = x$ and $f^n(y) \to \mathcal{O}(x)$ for Leb-a.e. y*
2. Absolutely continuous invariant measure: $\mu \ll \text{Leb}$, $\mu \circ f^{-1} = \mu$, *and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k y) = \int \varphi(x)\, d\mu(x)$$

for Leb-a.e. y and every $\varphi \in C([0,1])$

$S = \{\lambda \in [3,4] \mid \text{periodic attractor}\}$ (stable behaviour)

$U = \{\lambda \in [3,4] \mid \text{acim}\}$ (unstable behaviour)

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1. Attracting periodic orbit: $f^p(x) = x$ *and* $f^n(y) \to \mathcal{O}(x)$ *for Leb-a.e.* y
2. Absolutely continuous invariant measure: $\mu \ll \text{Leb}$, $\mu \circ f^{-1} = \mu$, *and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k y) = \int \varphi(x)\, d\mu(x)$$

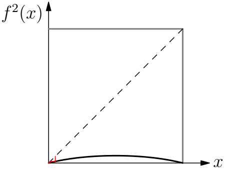
for Leb-a.e. y *and every* $\varphi \in C([0,1])$

$$S = \{\lambda \in [3,4] \mid \text{periodic attractor}\} \qquad \text{(stable behaviour)}$$
$$U = \{\lambda \in [3,4] \mid \text{acim}\} \qquad \text{(unstable behaviour)}$$

- S is open and dense... complement is a Cantor set
- U has positive Lebesgue measure despite being nowhere dense

Bifurcations

- $\lambda < 3$: one attracting fixed point, no periodic orbits
- $3 < \lambda < 3 + \epsilon$: fixed point is repelling, period-2 orbit is attracting



$\lambda = 0.5$

There is a bifurcation at $\lambda = 3$ – qualitative behaviour changes

Another bifurcation happens at $\lambda \approx 3.45$:

- Period-2 orbit becomes unstable
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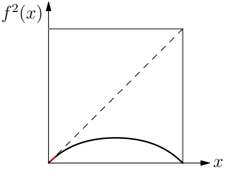

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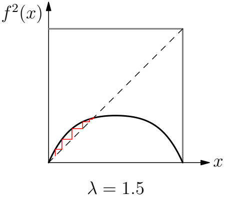

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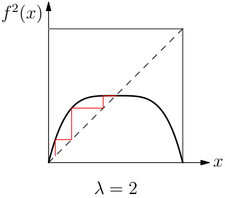

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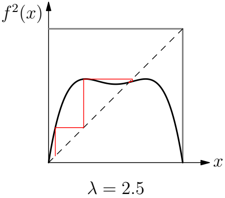

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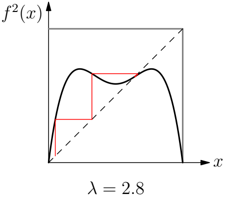

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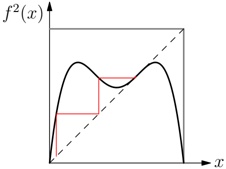

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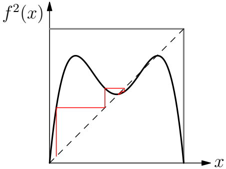

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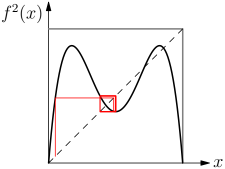

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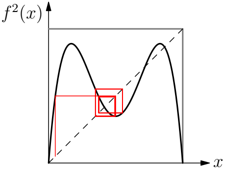

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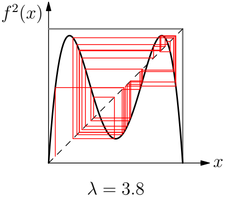

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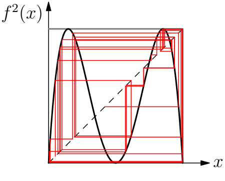

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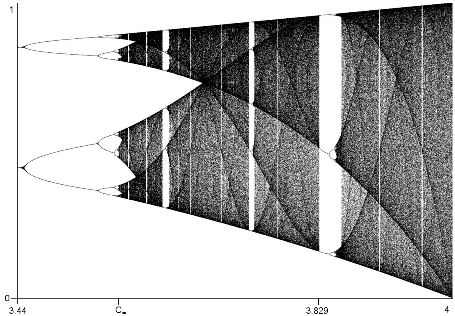
Period doubling cascades and universality



At λ_n, period 2^n orbit becomes unstable, period 2^{n+1} orbit is born: this is a **Period doubling cascade**

$\lambda_n \to \lambda_\infty \approx 3.569946\ldots$

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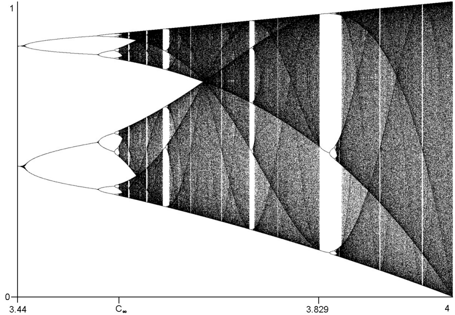

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$\lambda_n \to \lambda_\infty \approx 3.569946\ldots$

It turns out that $\lambda_\infty - \lambda_n \approx C\delta^n$, where $\delta \approx 1/4.6692\ldots$ is the Feigenbaum constant.

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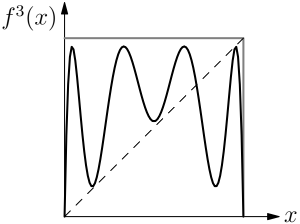
This applies to a very large class of one-parameter families f_λ, not just the logistic maps.

Windows of stability

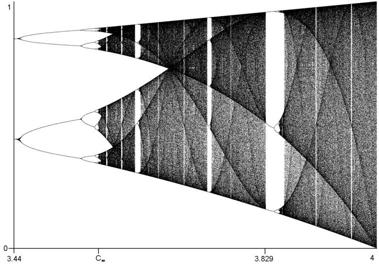


$\lambda = 3.815$

Contraction beats expansion for $\lambda < \lambda_\infty$.

- What happens for $\lambda > \lambda_\infty$?

Sometimes expansion wins (there is an acip and chaos), but there are windows of stability where f_λ has an attracting periodic orbit.



These windows of stability are dense in $[0, 4]$

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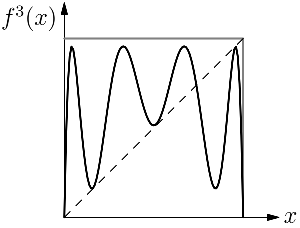

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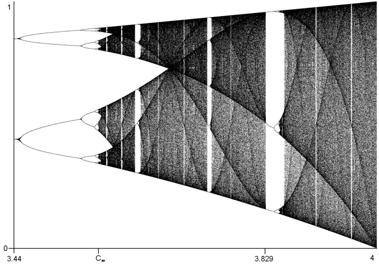

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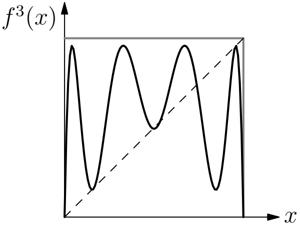

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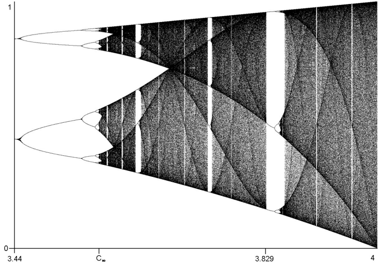

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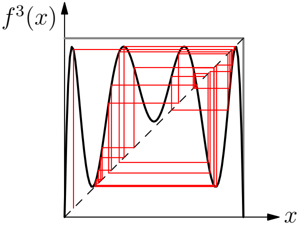

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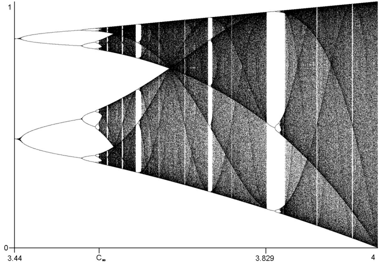

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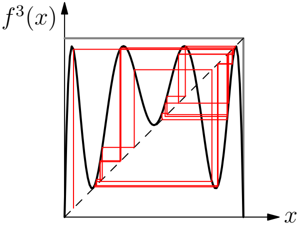

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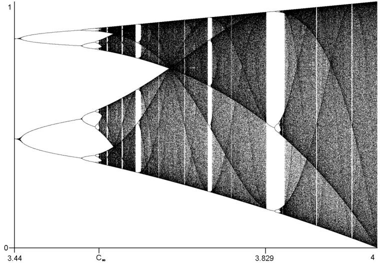

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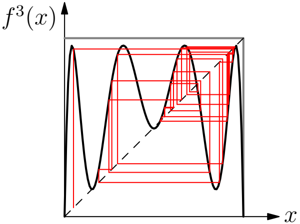

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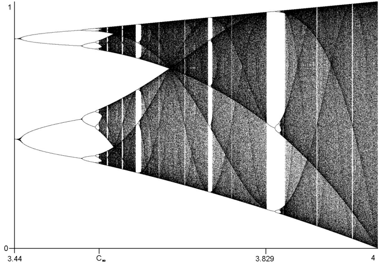

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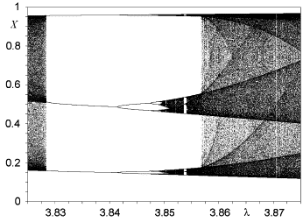
More on windows of stability

Periodic orbits appear in an order given by the Sharkovsky ordering:

$$1 \prec 2 \prec 4 \prec 8 \prec 16 \prec \cdots$$
$$\cdots$$
$$\cdots \prec 7 \cdot 2^n \prec 5 \cdot 2^n \prec 3 \cdot 2^n$$
$$\cdots$$
$$\cdots \prec 7 \cdot 2 \prec 5 \cdot 2 \prec 3 \cdot 2$$
$$\cdots \prec 7 \prec 5 \prec 3$$



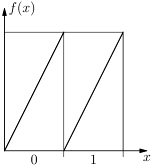
Each window of stability has its own period doubling cascade. *Self-similarity – a fractal sort of behaviour*

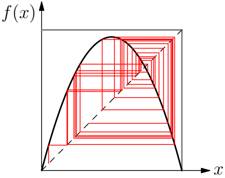
Universality constants are same as before.

Types of chaotic behaviour

Uniform expansion *(doubling map)*:

- Phase space expanded at every point
- Along an orbit, expansion at every time
- Stable under perturbations





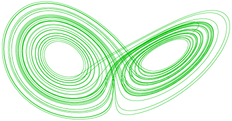
$\lambda = 3.8$

Non-uniform expansion *(logistic map)*:

- Some expansion, some contraction
- Along an orbit, contraction may occur but expansion wins asymptotically
- Very sensitive to perturbations

Higher dimensional *(Lorenz equations)*:

- Some directions expand and others contract
- Expansion and contraction may be uniform or non-uniform



Higher dimensions

Mechanism for chaos is **stretching** and **folding** of the phase space.

Formally, given a diffeomorphism $f\colon M \circlearrowleft$ of a smooth Riemannian manifold M, need a splitting of the tangent bundle:

$$T_x M = E^u(x) \oplus E^s(x)$$

- Invariance: $Df_x E^u(x) = E^u(f(x))$ and $Df_x E^s(x) = E^s(f(x))$
- Expansion in $E^u(x)$ and contraction in $E^s(x)$

Key step: Integrate $E^{s,u}$ to stable and unstable manifolds $W^{s,u} \subset M$.