Thermodynamic formalism for dynamical systems

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The talk in one slide

PHENOMENON	Deterministic systems can exhibit stochastic behaviour
MECHANISM	Driven by expansion + recurrence in phase space
IDEA	Treat as stochastic process; choose invariant measure. Given by equilibrium state in thermodynamic formalism
CHALLENGE	Mechanisms driving stochasticity may not be uniform

Predictions in dynamical systems

Key objects:

- X = phase space for a dynamical system.
  Points in X correspond to configurations of the system.
- $f: X \circlearrowleft$ describes evolution of the state of the system over a single time step. $f^n = f \circ \cdots \circ f$ *(n times)*

Standing assumptions: X is a compact metric space, f is continuous
Often X a smooth manifold, f a diffeomorphism

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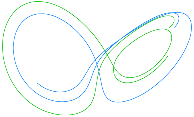
Common phenomenon: $\operatorname{diam} f^n(U)$ becomes large relatively quickly no matter how small U is. Stronger phenomenon:

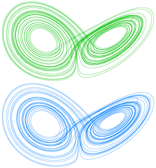
- iterates $f^n(U)$ become dense in X $\leftarrow$ *mechanism for rigorous results*

Examples

Lorenz equations (1963) – atmospheric dynamics

$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$
$$\dot{y} = x(\rho - z) - y \qquad \rho = 28$$
$$\dot{z} = xy - \beta z \qquad \beta = 8/3$$





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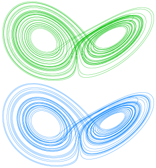

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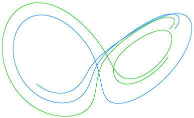
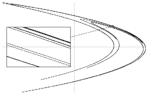


Hénon map (1976) – models stretching and folding



$$f(x, y) = (y + 1 - ax^2, bx) \qquad a = 1.4, b = .3$$

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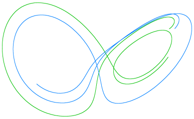

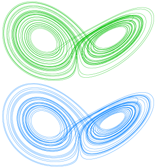

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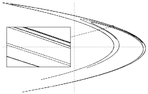

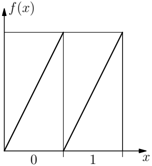

Lorenz and Hénon systems are non-uniformly hyperbolic.

Situation simplifies for (less realistic) uniformly hyperbolic systems, exemplified by the

Doubling map $f: S^1 \circlearrowleft, x \mapsto 2x \pmod 1$

Invariant and ergodic measures

Given $\varphi \in C(X)$, view $\varphi \circ f^n \colon X \to \mathbb{R}$ as sequence of random variables

- Pick $\mu \in \mathcal{M} = \{\text{Borel probability measures on } X\}$
- $(X, \mu, \varphi \circ f^n)$ defines a stochastic process

Does this process satisfy any limit laws? It is not usually i.i.d.

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$\mu \in \mathcal{M}$ is invariant if $\int \varphi \, d\mu = \int \varphi \circ f \, d\mu$ for all $\varphi \in C(X)$

- Equivalent to the RVs $(X, \mu, \varphi \circ f^n)$ being identically distributed
- $\mathcal{M}_f = \{\text{invariant measures}\} \subset \mathcal{M}$ *(convex, weak*-compact)*
- $\mathcal{M}_f^e = \{\text{extreme points of } \mathcal{M}_f\} = \{\text{ergodic measures}\}$

Each $\mu \in \mathcal{M}_f$ is a convex combination of ergodic measures (uniquely)

Limit laws

Theorem (G.D. Birkhoff, 1931)

If $\mu \in \mathcal{M}_f^e$ then $\frac{1}{n}\sum_{k=0}^{n-1}\varphi \circ f^k(x) \to \int \varphi \, d\mu$ for μ-a.e. x

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Other limit laws? CLT? Large deviations? Iterated logarithm?

- Identically distributed (by invariance) but generally **not** independent.

What ergodic measure should we use?

- Natural measure for diffeos is 'physical': volume. Often not invariant.

An abundance of measures

$\mathcal{M}_f^e$ is often very large.

- Example: $X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, $f = \sigma \colon x_0x_1x_2\ldots \mapsto x_1x_2x_3\ldots$.

Periodic measures: $f^p(x) = x \rightsquigarrow \mu = \frac{1}{p}(\delta_x + \delta_{fx} + \cdots + \delta_{f^{p-1}x}) \in \mathcal{M}_f^e$

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$\alpha, \beta > 0$, $\alpha + \beta = 1 \rightsquigarrow (\alpha, \beta)$-Bernoulli measure:

- $w \in \{0,1\}^n \rightsquigarrow$ cylinder set $[w] = \{x \in X \mid x_1 \cdots x_n = w\}$
- $k = \#$ of 0's in $w \Rightarrow \mu([w]) = \alpha^k \beta^{n-k}$

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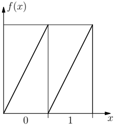
Also have Markov measures, Gibbs measures, etc.

How do we pick a good ergodic measure?

- (and what statistical properties does it have?)

Coding by symbolic systems



Doubling map $f\colon S^1 \circlearrowleft$, $x \mapsto 2x \pmod 1$

Full shift $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, $f = \sigma\colon x_0x_1x_2\ldots \mapsto x_1x_2x_3\ldots$

General procedure for symbolic description of dynamics:

1. Partition X as a disjoint union $A_1 \cup \cdots \cup A_d$
2. $f^n(x) \in A_{y_n}$ defines $y = \pi(x) \in \{1,\ldots,d\}^{\mathbb{N}}$
3. $\pi\colon X \to \{1,\ldots,d\}^{\mathbb{N}}$ is the coding map
4. $Y = \overline{\pi(X)}$ is the coding space

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi\downarrow & & \downarrow\pi \\ Y & \xrightarrow{\sigma} & Y \end{array}$$

If $y_1\ldots y_n = y_1'\ldots y_n'$ but $y_{n+1} \neq y_{n+1}'$, then $d(y,y') = 2^{-n}$

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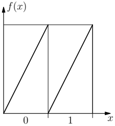

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Coding space is closed and σ-invariant: $\sigma(Y) \subset Y$.

Typically many "forbidden" sequences. When is Y "good"?

Entropy for shift spaces

Topological entropy of a shift space X:

- $\mathcal{L} = \{\text{words that appear in some } x \in X\}$ = language of X
- $h_{\text{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n$ $\quad \mathcal{L}_n = \{\text{words of length } n\} \subset \mathcal{L}$

Example

$X = \Sigma_2^+ \quad \Rightarrow \quad \#\mathcal{L}_n = 2^n \quad \Rightarrow \quad h_{\text{top}}(X) = \log 2$

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Measure-theoretic entropy for $\mu \in \mathcal{M}_f$:

- $h(\mu) := \lim_{n\to\infty} \frac{1}{n} \sum_{w\in\mathcal{L}_n} H(\mu[w])$ $\quad H(p) = -p \log p$

Example

Entropy of (α, β)-Bernoulli measure is $h(\mu) = -\alpha \log \alpha - \beta \log \beta$.

Variational principles

Variational principle: $h_{\mathrm{top}}(X) = \sup\{h(\mu) \mid \mu \in \mathcal{M}_f\}$

- $h(\mu) = h_{\mathrm{top}}(X) \rightsquigarrow \mu$ is a measure of maximal entropy (MME)

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Generalises to topological pressure of a potential function $\varphi \in C(X)$:

- $\Lambda_n(\varphi) = \sum_{w \in \mathcal{L}_n} e^{S_n\varphi(w)}$ $\qquad S_n\varphi(w) = \sup_{x \in [w]} \sum_{k=0}^{n-1} \varphi(\sigma^k x)$
- Topological pressure of φ is $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(\varphi)$
- $P(\varphi) = \sup\{h(\mu) + \int \varphi\, d\mu \mid \mu \in \mathcal{M}_f\}$
- A measure achieving the supremum is an equilibrium state

Example: $X = \Sigma_2^+$, $\varphi(x) = s\chi_{[0]} + t\chi_{[1]}$

- $P(\varphi) = \log(e^s + e^t)$, unique eq. state is $(e^{s-P(\varphi)}, e^{t-P(\varphi)})$-Bernoulli

Unique equilibrium states

Unique equilibrium states often have strong statistical properties: central limit theorem, decay of correlations, large deviations, etc.

- the sequence of observations $(X, \mu, \varphi \circ f^n)$ has many properties in common with i.i.d. sequence of random variables

Decay of correlations:

- $\varphi, \psi \in C^\alpha + \int \varphi \, d\mu = 0 \Rightarrow C_n(\varphi, \psi) = \int (\varphi \circ f^n)\psi \, d\mu \to 0$
- Often: unique $\Rightarrow$ exponential, non-unique $\Rightarrow$ polynomial.

Central limit theorem:

- $\psi \in C^\alpha + \int \psi \, d\mu = 0 \Rightarrow \exists \xi \geq 0$ such that for all $r \in \mathbb{R}$,

$$\mu \left\{ x \mid \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \psi(f^k x) < r \right\} \xrightarrow{n \to \infty} \frac{1}{\xi\sqrt{2\pi}} \int_{-\infty}^{r} e^{-x^2/2\xi^2} \, dx$$

SRB measures

Key example: f a diffeo, $TM = E^u \oplus E^s$ a Df-invariant splitting,

$$\|Df^n(v^u)\| \to \infty \text{ and } \|Df^n(v^s)\| \to 0 \text{ exponentially in } n.$$

Equilibrium states for $-\log|\det(Df|_{E^u})|$ are 'physical' measures.

- Not smooth on M, but smooth along unstable manifolds

Existence, exponential decay of correlations, CLT known in many cases

- Uniformly hyperbolic systems: (Ya. Sinai, D. Ruelle, R. Bowen)
- NUH systems: (Benedicks–Carleson–Young–Wang, Alves–Bonatti–Viana, C.–Dolgopyat–Pesin)

A (brief) digression: some applications

- **Hausdorff dimension:** If $f : M \circlearrowleft$ is conformal and J is a uniformly expanding repeller for f, then $\dim_H J = t$ solves $P_J(-t \log \|Df\|) = 0$ (R. Bowen 1979, D. Ruelle 1982). Also holds in more general settings (Gatzouras–Peres 1997, Rugh 2008, C. 2011).

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- **Biology:** pressure can be used to distinguish between coding and non-coding DNA sequences (D. Koslicki, D. Thompson)

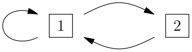
Subshifts of finite type

Unique MME for full shift is $(\frac{1}{2}, \frac{1}{2})$-Bernoulli

- Has exponential decay of correlations, CLT, large deviations

More general: $X \subset \{1, \dots, d\}^{\mathbb{N}}$ is a subshift of finite type (SFT)

- Set of walks on a directed graph with vertices labeled $1, \dots, d$.



$X = \{\text{words on } \{1, 2\} \text{ such that 2 never follows 2}\}$

Given by $d \times d$ transition matrix A

- $A_{ij} = 1$ if j can follow i, and 0 otherwise
- $\lambda =$ largest eigenvalue of $A \Rightarrow h_{\text{top}}(X, f) = \log \lambda$
- Unique MME given in terms of left and right eigenvectors for λ

Uniformly hyperbolic systems

Results generalise to equilibrium states for Hölder potentials φ

- $\varphi = 0$: transition matrix $A\colon \mathbb{R}^d \to \mathbb{R}^d$ contracts positive cone
- More generally: Perron–Frobenius operator $L_\varphi\colon C^\alpha(X) \to C^\alpha(X)$

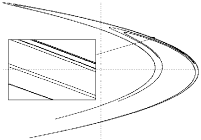
A diffeomorphism $f\colon M \to M$ is uniformly hyperbolic if there is a Df-invariant splitting $T_xM = E^u(x) \oplus E^s(x)$ and $\chi > 1$ such that

- $\|Df(v^u)\| > \chi\|v^u\|$
- $\|Df(v^s)\| < \chi^{-1}\|v^s\|$

Uniformly hyperbolic systems have Markov partitions

- Can be coded using SFTs
- Unique equilibrium states with strong statistical properties

Non-uniform hyperbolicity



Many (most) natural "chaotic" systems are not uniformly hyperbolic...

Hénon map

- $E^u(x)$ and $E^s(x)$ depend only measurably on x, and may become arbitrarily close together
- $\|Df^n(v^s)\| \leq C_x \chi^{-n} \|v^s\|$ and $\|Df^n(v^u)\| \geq C_x^{-1} \chi^n \|v^u\|$, but C_x depends only measurably on x, and may become arbitrarily large

Cannot be coded with SFTs. Need to consider broader classes of symbolic systems in order to study non-uniform hyperbolicity.

- One possibility: use a countable alphabet
- Another option: finite alphabet, but more general language

Multiple MMEs

Beyond SFTs, what classes of symbolic systems have unique MMEs?

- Should be transitive (any two words can eventually be joined): otherwise consider $\{1,2\}^{\mathbb{N}} \cup \{1,2\}^{\mathbb{N}}$. Has $h_{\text{top}} = \log 2$ and two MMEs: ν on $\{1,2\}^{\mathbb{N}}$ and μ on $\{1,2\}^{\mathbb{N}}$, both $(\frac{1}{2}, \frac{1}{2})$-Bernoulli

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Need more than transitivity: $X \subset \Sigma_5 = \{0,1,2,1,2\}^{\mathbb{N}}$. Define the language $\mathcal{L}$ by $v0^n w,\ w0^n v \in \mathcal{L}$ if and only if $n \geq |v| + |w|$.

- Transitive and $h_{\text{top}}(X,\sigma) = \log 2$
- Same two measures of maximal entropy as above

Uniform transitivity

Full shift: words can be freely concatenated: $v, w \in \mathcal{L} \Rightarrow vw \in \mathcal{L}$

Transitive $\Rightarrow \forall v, w \in \mathcal{L}$ there exists $u \in \mathcal{L}$ such that $vuw \in \mathcal{L}$

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- Specification: $\exists \tau$ such that $|u| \leq \tau$ for all v, w

Transitive SFTs have specification

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Transitive SFTs have specification

Theorem (R. Bowen, 1974)

Specification $\Rightarrow$ unique equilibrium state μ_φ for Hölder potential φ

Theorem (C., 2013)

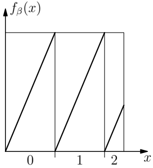
μ_φ has exponential decay of correlations and satisfies the CLT

β-shifts

For $\beta > 1$, Σ_β is the coding space for the map

$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^\infty a_n \beta^{-n}$



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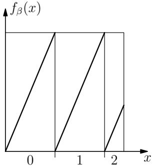
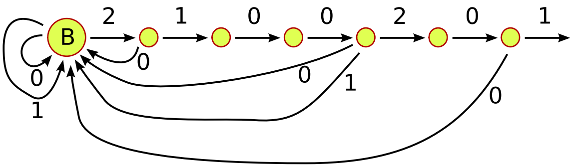


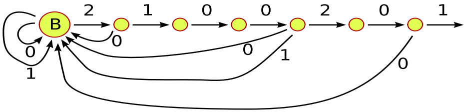
$x \in \Sigma_\beta \quad \Leftrightarrow \quad x$ labels a walk starting at **B** $\quad \Leftrightarrow \quad \sigma^n x \preceq 1_\beta$ for all n

(Here $1_\beta = 2100201\dots$)



Towers

Specification fails if 1_β contains arbitrarily long strings of 0's



Still get unique ES for Lipschitz φ (P. Walters 1978, F. Hofbauer 1979)

Σ_β given by a countable graph $\Rightarrow$ use countable state analogue of SFTs

Towers

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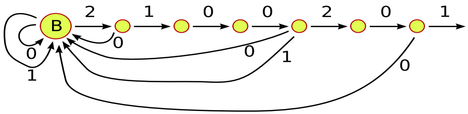

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Σ_β given by a countable graph $\Rightarrow$ use countable state analogue of SFTs

This leads to tower approach to non-uniform hyperbolicity

- Idea: Find $Z \subset X$ and a countable partition $Z = \bigsqcup_i Z_i$ such that $f^{\tau_i}(Z_i) = Z$ for some inducing time τ_i
- Z "big enough" $+ \tau_i$ "small enough" $\Rightarrow$ unique ES, stat. properties

Used for Hénon maps and billiard systems (Young 1998)

Decompositions

When is it possible to build a tower? Or to get results via other means?

For symbolic systems, can use decompositions of the language $\mathcal{L}$.

$\mathcal{L} = \mathcal{S}\mathcal{G}\mathcal{S} \quad \Leftrightarrow$ $\mathcal{G}, \mathcal{S} \subset \mathcal{L}$ are such that every $w \in \mathcal{L}$ can be written as $w = v^p u v^s$ for some $u \in \mathcal{G}$ and $v^p, v^s \in \mathcal{S}$

Example

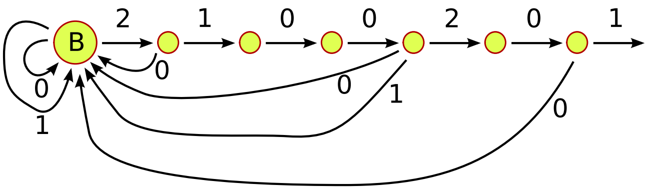
$X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ $\quad \mathcal{G} = \{1w1 \mid w \in \mathcal{L}\}$ $\quad \mathcal{S} = \{0^n \mid n \geq 0\}$

- The entropy of $\mathcal{S}$ is $h(\mathcal{S}) = \overline{\lim}_{n\to\infty} \frac{1}{n} \log \#\mathcal{S}_n$

Key observation: If $\mathcal{G}$ has specification and $h(\mathcal{S}) < h_{\mathrm{top}}(X)$, then many ideas from Bowen's proof can be adapted.

For the full shift, this is unnecessary, since $\mathcal{L}$ already has specification, but the above decomposition is useful for other reasons.

Non-uniform specification for Σ_β



The only obstruction to specification is the tail of the sequence 1_β.

Let $\mathcal{G} = \{$words whose path begins and ends at **B**$\}$

- $\mathcal{G}$ has specification

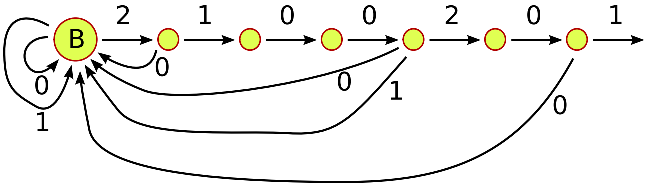
Let $\mathcal{S} = \{$words whose path never returns to **B**$\}$ *(cusp excursions)*

- $\mathcal{L} = \mathcal{G}\mathcal{S}$ and $h(\mathcal{S}) = 0$

Obstructions to specification

$\mathcal{G} \subset \mathcal{L} \rightsquigarrow \mathcal{G}^M := \{uvw \mid v \in \mathcal{G}, |u|, |w| \leq M\} \rightsquigarrow$ filtration $\mathcal{L} = \bigcup_M \mathcal{G}^M$

For the β-shift, $\mathcal{G}^M$ corresponds to walks ending on one of the first M vertices. Can return from these vertices to the base vertex in uniform time, so each $\mathcal{G}^M$ has specification.



"Every $\mathcal{G}^M$ has specification" means we can glue words together, provided we are allowed to remove an obstructing piece from the end of each word.

Equilibrium states with non-uniform specification

Theorem (C.–Thompson, 2013)

Let X be a symbolic system with language $\mathcal{L}$. Suppose $\mathcal{L}$ has a decomposition $\mathcal{S}\mathcal{G}\mathcal{S}$ such that every $\mathcal{G}^M$ has specification. If φ is Hölder and $P(\mathcal{S},\varphi) < P(X,\varphi)$, then φ has a unique equilibrium state μ_φ.

$$P(\mathcal{S},\varphi) = \overline{\lim}_{n\to\infty} \frac{1}{n} \log \Lambda_n(\mathcal{S}_n,\varphi)$$

Theorem (C., 2013)

Under the above conditions, there is a tower such that $\mu_\varphi\{x \mid \tau(x) \geq n\}$ decays exponentially in n. In particular, μ_φ has exponential decay of correlations and satisfies the CLT.

Large deviations

Given μ and φ, let $LD_n(\epsilon) = \{x \in X \mid |\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k x) - \int \varphi\, d\mu| > \epsilon\}$

Birkhoff ergodic theorem $\Rightarrow$ $\mu(LD_n(\epsilon)) \to 0$ as $n \to \infty$

Question: how quickly does $\mu(LD_n(\epsilon))$ decay?

μ satisfies large deviations principle (LDP) with rate function $q(\epsilon)$ if $\lim_{n\to\infty} \frac{1}{n}\log(LD_n(\epsilon)) = q(\epsilon) < 0$

- Specification $\Rightarrow$ μ_φ has LDP (Young, 1990)
- Non-uniform ($\mathcal{SGS}$) specification $\Rightarrow$ μ_φ has LDP if $\mathcal{L}$ is edit approachable by $\mathcal{G}$ (C.–Thompson–Yamamoto, 2013)

Edit approachable: $w \in \mathcal{L}_n$ can be turned into $\tilde{w} \in \mathcal{G}$ by making $o(n)$ edits

Non-symbolic applications

All the results quoted using specification are in the symbolic setting.

This is a playground motivating results for smooth systems.

Uniqueness results have been extended to smooth systems assuming non-uniform version of expansivity.

Currently being developed: Applications to partially hyperbolic systems, geodesic flows on manifolds of non-positive curvature.