UFRJ – Probability in Dynamics

Specification, hyperbolicity, and towers

Vaughn Climenhaga

Setting: X cpt met sp, $f: X \circlearrowleft$ cts
$\mathcal{M}_f = \{\text{Borel } f\text{-inv. prob}\}$
$\varphi: X \to \mathbb{R}$ cts

$\Bigg\}$ equilibrium state maximises $h_\mu(f) + \int \varphi \, d\mu$ over $\mathcal{M}_f$

Questions: ⓪ existence ① uniqueness ② statistical properties

Answers under certain conditions on X, f, φ.

	(X, f)	φ	
⓪	shift space on finite alphabet / expansive	cts	$\Rightarrow$ existence
①	~~[illegible]~~ specification (uniform mixing)	Hölder/Bowen	$\Rightarrow$ uniqueness
②	SFT / Axiom A	Hölder	$\Rightarrow$ EDC + CLT

What about non-uniform hyperbolicity? (non-uniform mixing)

Consider two examples:

$\boxed{\text{I}}$ S-gap shifts. $X \subset \{0,1\}^{\mathbb{Z}}$ determined by $S \subset \mathbb{N}$: 10^n1 allowed $\Leftrightarrow n \in S$. (infinite)

$\boxed{\text{II}}$ partially hyperbolic on $\mathbb{T}^3$: Mañé's example (DA)
perturb f_A, where $A \in SL(3, \mathbb{Z})$ has $\lambda_1 < \lambda_2 < 1 < \lambda_3$,
by a pitchfork bifurcation

(also Bonatti–Viana version)

Summary of results: $\boxed{\text{I}}$ already understood using towers, we use it as a "sandbox" to study non-uniform expansivity & specification, eventually applying to $\boxed{\text{II}}$.

(NB) neither $\boxed{\text{I}}$ nor $\boxed{\text{II}}$ has specification. (assume S has unbdd gaps)

Known results

How to study S-gap shift? Embed $S^{\mathbb{Z}}$ (full shift on ctbl alphabet)

$$\cdots n_{-2}\, n_{-1} \,.\, n_0 \,.\, n_1\, n_2 \cdots \longmapsto \cdots 1 0^{n_{-1}} 1 0^{n_0} 1 0^{n_1} 1 \cdots$$

Relate ES for (X, φ) to ES for $(S^{\mathbb{Z}}, \Phi)$, use ~~thermodynamic~~ formalism for ctbl state shifts, conclude ⓪, ①, ② whenever φ Hölder.

Key tool: ~~$\mathcal{F} = \{10^n \mid n \in S\}$ satisfies~~ $\mathcal{F} = \{10^{n_1}10^{n_2}\cdots 0^{n_k} \mid n_i \in S\}$

$\boxed{I_0}$ $\quad v, w \in \mathcal{F} \Rightarrow vw \in \mathcal{F}$

and $\mathcal{F}$ is "large enough".

Rmk: This "tower approach" is very powerful, but for non-symbolic systems building a tower is difficult. Specification seems easier to verify.

New results

$\mathcal{G} \subset \mathcal{L} = \mathcal{L}(X)$ has specification if

$\boxed{I}$ $\quad \exists\, \tau \in \mathbb{N}$ s.t. $\forall\, v, w \in \mathcal{G}$ $\exists\, u \in \mathcal{L}$, $|u| \le \tau$, $vuw \in \mathcal{G}$

Consider "pressure of collection of words" $\mathcal{D} \subset \mathcal{L}$:

$$P(\mathcal{D}, \varphi) = \varlimsup_{n\to\infty} \frac{1}{n} \log \sum_{w \in \mathcal{D}_n} \sup_{x \in [w]} e^{S_n \varphi(x)}$$

$P(\mathcal{L}, \varphi)$ gives usual top pressure.

Need a decomposition $\mathcal{L} \subset \mathcal{C}^p \mathcal{G} \mathcal{C}^s$, ie.

$\forall\, w \in \mathcal{L}$ $\exists\, u^{p,s} \in \mathcal{C}^{p,s}$ & $v \in \mathcal{G}$ s.t. $w = u^p v u^s$

$\boxed{II}$ $\quad \exists\, \mathcal{L} \subset \mathcal{C}^p \mathcal{G} \mathcal{C}^s$ s.t. $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(\mathcal{L}, \varphi)$.

Also need $\boxed{III}$ $\quad \sup_{w \in \mathcal{G}} \sup_{x, y \in [w]} |S_n\varphi(x) - S_n\varphi(y)| < \infty$

$\boxed{\text{Rmk}}$ $\varphi = 0$, (II) reduces to entropy, check for S-gap $h(\mathcal{C}^p \cup \mathcal{C}^s) = 0$.

$\boxed{\text{Thm}}$ (C. - Thompson)

(I^+)–$(III) \Rightarrow \exists!$ ES

Also for S-gap, φ Hölder $\Rightarrow h(ES) > 0$

Pf is ad hoc. Can it be made general, beyond S-gap & β?

Thm (C. - Thompson)

same in non-symbolic setting (need NU expansivity)

$$P^{\perp}_{s,B}(\varphi) < P(\varphi) \quad \& \quad P^{\perp}_{exp}(\varphi) < P(\varphi) \Rightarrow \exists! \text{ ES}$$

Thm (C. - Fisher - Thompson)

applies to Mañé example

($\forall$ Hölder φ $\exists$ C^1-open set of g s.t. $\exists!$ ES)

What about statistical properties?

* No proof in literature even w/ uniform specification
* Well-known using SFT / ctbl Markov shift

$\Rightarrow \boxed{I_0} + \boxed{II} + \boxed{III}$ enough

Question Can we go from $\boxed{I} \Rightarrow \boxed{I_0}$?

A. Bertrand, 1988: If $\mathcal{L}$ has spec then $\exists$ $s \in \mathcal{L}$ synchronizing

$(vs, sw \in \mathcal{L} \Rightarrow vsw \in \mathcal{L})$

Interpretation: expansive + spec $\Rightarrow$ local product structure somewhere

Then $\mathcal{B} = \{v \mid svs \in \mathcal{L}\} \Rightarrow \mathcal{F} = s\mathcal{B}$ has $\boxed{I_0}$

$\boxed{\text{Thm}}$ expansive + specification $\Rightarrow$ tower $\Rightarrow$ $\exists!$ ES $\boxed{+ \text{ stat prop}}$

What about non-uniform spec? Need 1 more condition:

$\boxed{IV}$ If $uvw \in \mathcal{L}$, $uv \in \mathcal{G}$, $vw \in \mathcal{G}$, ~~then~~ $uvw \in \mathcal{G}$

Rmk Natural for PH example

$\boxed{\text{Thm}}$ (C.)

- If $\mathcal{G}$ has $\boxed{I}$ then $\exists$ $r \in \mathcal{G}$, $c \in \mathcal{L}$ s.t. $s = rcr \in \mathcal{G}$,
  $\mathcal{B} = \{w \in \mathcal{L} \mid rwr \in \mathcal{G}\}$, $\mathcal{F} = s\mathcal{B}$ $\Rightarrow$ $\mathcal{F}$ has $\boxed{I_0}$
- If $\mathcal{G}$ has $\boxed{II}, \boxed{III}, \boxed{IV}$, so does $\mathcal{F}$: $\exists$ $\mathcal{E}^{p,s}$ s.t.
  $\mathcal{L} \subset \mathcal{E}^p \mathcal{F} \mathcal{E}^s$ & $P(\mathcal{E}^p \cup \mathcal{E}^s, \varphi) < P(\varphi)$
- $\boxed{I}$-$\boxed{IV}$ $\Rightarrow$ $\exists!$ ES, ~~tower~~ w/ exp ~~tails~~, EDC, CLT

Q Non-symbolic version of ~~this~~?