The bigness of things

Vaughn Climenhaga

University of Houston

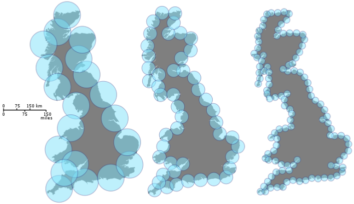
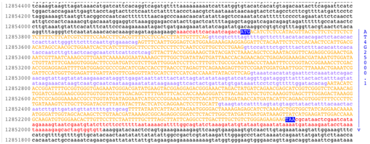


Image from Wikipedia



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How big is it?

Meaning of “big” depends on what “it” is, and why we care.

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a crowd of people?

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a crowd of people? number weight

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How big is . . .

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the internet?	# of websites	data	useful data

Various notions of “bigness”

Concrete, familiar meanings of “big” from the previous slide:

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1. length
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More abstract meanings: “amount of data”?

- We are used to thinking of kB, MB, GB, TB, etc.
- But a 500 GB hard drive where every bit is set to ‘0’ doesn’t have much data on it. . .

Subsets of $\mathbb{R}^3$

Focus on familiar meanings for now. Consider some subsets of $\mathbb{R}^3$.

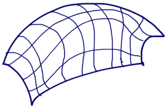
(a) finite set

0-dimensional

(b) curve

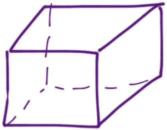
1-dimensional

(c) surface



2-dimensional

(d) open region



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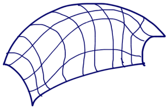

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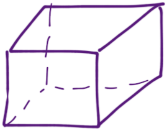

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Cardinality:

- (a): Good way to measure how big a finite set is
- (b)–(d) have infinite cardinality

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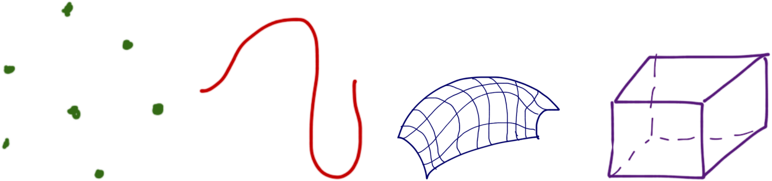

0-dimensional 1-dimensional 2-dimensional 3-dimensional

Length:

- (a) has zero length. (Cover each point with tiny intervals)
- (b): Good way to measure how big a curve is
- (c)–(d) have infinite length: no curve of finite length can cover

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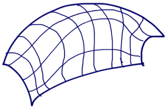

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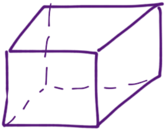

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Area:

- (a)–(b) have zero area. (Cover with tiny discs)
- (c): Good way to measure how big a surface is
- (d) has infinite area

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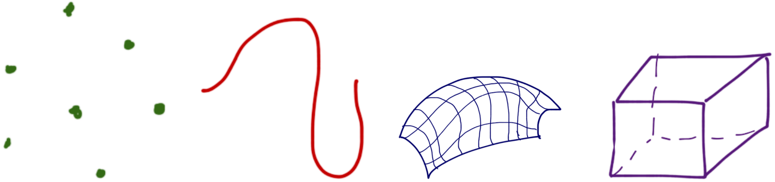

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Volume:

- (a)–(c) have zero volume
- (d): Good way to measure how big an open region is

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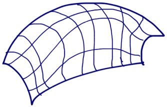

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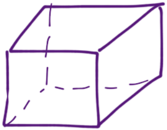

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Moral: To say how "big" a thing is, need to know its dimension.

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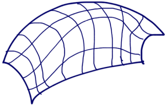

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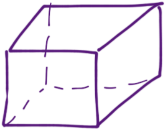


3-dimensional

Moral: To say how "big" a thing is, need to know its dimension.

- Dimension itself is a notion of bigness
- What is "dimension"? Seems to be which measure we use...

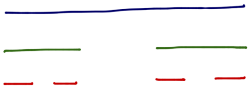
Example 1: A Cantor set

Consider the sets

$C_0 = [0, 1]$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$



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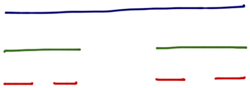

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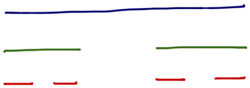

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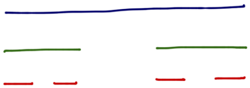

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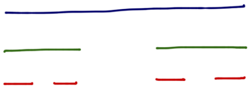

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What is the dimension of C? Between 0 and 1.

Example 2: The Koch curve

Consider the curves

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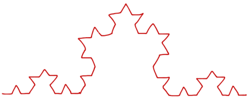
$K_2 =$ 

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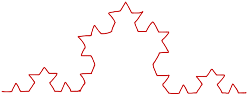
$K_2 =$ 

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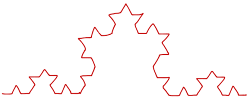
$K_2 =$ 

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Fact 1: K has infinite length. (Length of K_n is $4^n 3^{-n}$)

Fact 2: K has zero area. (Exercise – cover it with small rectangles)

What is dimension?

Algebraic idea: # of parameters/coordinates. (Always an integer!)

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More geometric idea: dimension is a scaling exponent.

Given $\lambda > 0$ and $E \subset \mathbb{R}^3$, let $\lambda E = \{\lambda \mathbf{x} \mid \mathbf{x} \in E\}$

- $\mathrm{volume}(\lambda E) = \lambda^3 \cdot \mathrm{volume}(E)$
- $\mathrm{area}(\lambda E) = \lambda^2 \cdot \mathrm{area}(E)$
- $\mathrm{length}(\lambda E) = \lambda^1 \cdot \mathrm{length}(E)$
- $\mathrm{cardinality}(\lambda E) = \lambda^0 \cdot \mathrm{cardinality}(E)$

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"Correct" thing to do now is find for each $\alpha > 0$ a measure

$$\mu_\alpha \colon \{\text{subsets of } \mathbb{R}^3\} \to [0, \infty] \text{ such that } \mu_\alpha(\lambda E) = \lambda^\alpha \mu(E)$$

This is α-dimensional Hausdorff measure, but requires technicalities

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Previous slide highlighted self-similarity of measures.

Think about self-similarity of sets. Scale a set E by a factor of $\frac{1}{2}$. How many copies needed to recover original shape?

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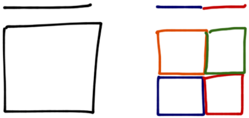

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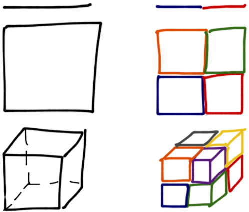
$E = [0,1]^2$ — $4 = 2^2$ copies

$E = [0,1]^3$ — $8 = 2^3$ copies



Moral: If E is a union of n copies of λE, then E is self-similar, and the dimension of E is α, where $n = \lambda^{-\alpha}$.

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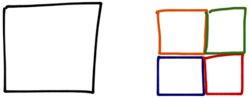

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Solve this to write $\dim E = \alpha = \frac{\log n}{-\log \lambda}$.

Examples

Apply the formula $\dim E = \frac{\log n}{-\log \lambda}$ to some examples.

E	λ	n	$\dim E$
interval	$\frac{1}{2}$	2	$\frac{\log 2}{\log 2} = 1$
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cube	$\frac{1}{2}$	8	$\frac{\log 8}{\log 2} = 3$
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May consider other Cantor sets: 

- scale by $\frac{1}{5}$, use 3 copies to build: $\dim = \frac{\log 3}{\log 5}$

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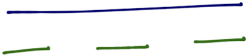
May consider other Cantor sets: 

- scale by $\frac{1}{5}$, use 3 copies to build: $\dim = \frac{\log 3}{\log 5}$

What about something like 

?

Dimension as a growth rate

Alternate way to derive dimension of our examples:

1. Given $r > 0$, break set into pieces of diameter $\leq r$
2. $N(r) =$ number of such pieces

Observe that $N(r) \approx r^{-\dim}$ $\left\{ \right.$

- interval: $N(r) = r^{-1}$
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Conclusion: $\dim = \lim_{r \to 0} \frac{\log N(r)}{-\log r}$ (growth rate of $N(r)$)

- Cantor set: $N(3^{-n}) = 2^n$, so $\frac{\log N(3^{-n})}{-\log(3^{-n})} = \frac{\log 2}{\log 3}$
- Koch curve: $N(3^{-n}) = 4^n$, so $\frac{\log N(3^{-n})}{-\log(3^{-n})} = \frac{\log 4}{\log 3}$

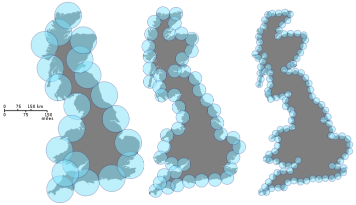
More general examples

Coastline of Britain

- r = size of ruler
- $rN(r)$ = measured length

$N(r) \approx r^{-1.25}$

Measured length $\approx r^{-.25} \to \infty$



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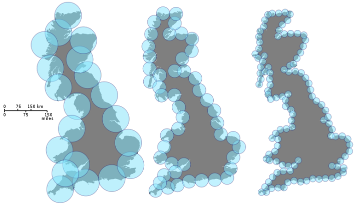

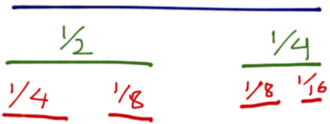

Can show that $N(2^{-(k+2)}) = F_k$, the kth Fibonacci number

Use fact that $F_k \approx \left(\frac{1+\sqrt{5}}{2}\right)^k$ to deduce that $\dim = \frac{\log(1+\sqrt{5})}{\log 2} + 1$

Bernoulli processes

Consider the following two stochastic processes:

1. Flip a coin repeatedly, write down outcome (H or T)
2. Roll a die repeatedly, write down the number from 1 to 6

Which one is "bigger"? The second one, but why?

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For n possible outcomes, need $2^h = n$, so $h = \log_2 n$.

- First process: $h = \log_2 2 = 1$
- Second process: $h = \log_2 6 \in (2, 3)$

$\boxed{h = \text{entropy}}$

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What if I use a weighted coin? Say $\mathbb{P}(H) = \frac{1}{3}$ and $\mathbb{P}(T) = \frac{2}{3}$.

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Coin with weights $\frac{1}{3}$ and $\frac{2}{3}$:

- H carries information $\log_2(3)$, and T carries info $\log_2(\frac{3}{2})$

Entropy $= \frac{1}{3}\log_2(3) + \frac{2}{3}\log_2(\frac{3}{2}) = \log_2(3) - \frac{2}{3} < 1$ (log is concave)

Maximising entropy

Suppose I use a coin with weights p and q.

$$\boxed{\begin{array}{l} p, q \in [0,1] \\ p + q = 1 \end{array}}$$

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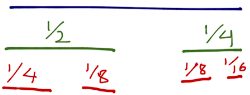
Because log is concave down we always have entropy ≤ 1

Strictly concave $\Rightarrow$ equality iff $p = q = \frac{1}{2}$

Relationship to dimension

Recall example of $(\frac{1}{2}, \frac{1}{4})$-Cantor set.



After n iterations, get a set C_n with 2^n intervals

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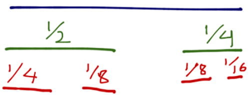

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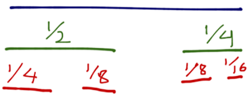

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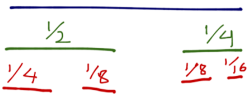

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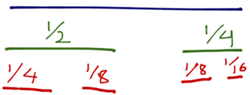

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Get actual dimension by maximising over (p, q).

Information compression

Entropy measures information content

- Related: how much can data be compressed?

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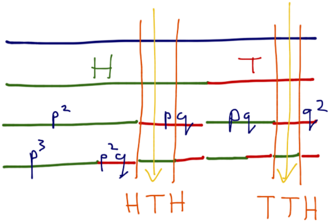
Shannon's source coding theorem:

If we run n iterates of a process (IID) with entropy h, the results can be stored in nh bits of information, but no fewer.



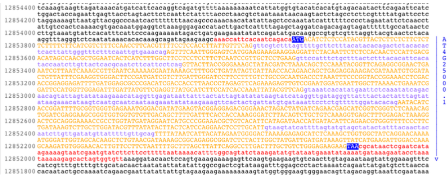
Idea: First n results determine a subinterval of $[0, 1]$

- "Typical" interval has width $p^{pn}q^{qn} = 2^{-nh}$
- Takes n bits to encode that much precision

Information content

Entropy can be used to analyse genetic data.

- Genome: string of symbols A, C, G, T
- Some regions more important than others



Topological entropy and topological pressure

- Quantities related to entropy discussed above
- Can be adapted to study genetic data
- High entropy/pressure $\Rightarrow$ high information content $\Rightarrow$ more likely to be a coding region of the genome