# Specification + towers (Maryland)

**Setting**

$X$ cpt met sp, $f : X \circlearrowleft$ cts, $\quad$ [ $\varphi : X \to \mathbb{R}$ cts OR $m \in \mathcal{M}$ a ref meas ]

$\mathcal{M}_f = \{$Borel $f$-inv. prob$\}$

Search for • equilibrium state (maximises $h_\mu(f) + \int \varphi \, d\mu$)

or • a.c.i.p. ($\mu \in \mathcal{M}_f$, $\mu \ll m$) $\quad$ (eg. SRB)

$\boxed{Q}$ ⓪ existence?

① uniqueness?

② properties?

For now let $X$ be a shift space, $\varphi$ Hölder

$\boxed{\text{FACT}}$ $X$ an SFT $\Rightarrow \exists!$ ES $\mu = \mu_\varphi$, and $\mu$ has

[EDC] exponential decay of correlations:

$\psi_1, \psi_2 \in C^\alpha \Rightarrow \left| \int (\psi_1 \circ \sigma^{n}) \psi_2 \, d\mu - \int \psi_1 \, d\mu \int \psi_2 \, d\mu \right| \leq C e^{-\delta n}$

[CLT] central limit theorem:

$\psi \in C^\alpha$, $\int \psi \, d\mu = 0 \Rightarrow \frac{1}{\sqrt{n}} S_n \psi \longrightarrow$ Normal in distribution

Smooth systems – same holds for mixing Axiom A. (unif hyp)

What is special about SFTs? (or Ax A?)

① Markov structure gives good control of $\mathcal{L}_\varphi$ (RPF operator)

$\to$ spectral gap, results from functional analysis

② specification property $\to$ uniform mixing

* ① gives $\exists$, $!$, EDC + CLT, $\quad$ ② only gives $\exists$, $!$

Given $X \subset A^{\mathbb{N}}$ closed, $\sigma$-inv, let $\mathcal{L}$ = language of $X$

(*) specification = $\exists\, \tau \in \mathbb{N}$ s.t. $\forall\, u, v \in \mathcal{L}$ $\exists\, w \in \mathcal{L}$ with $|w| \le \tau$ and $uwv \in \mathcal{L}$.

For non-uniformly hyperbolic systems, get $\exists + ! +$ EDC/CLT using inducing schemes / Young towers

* embed full shift on ctbl alphabet into system

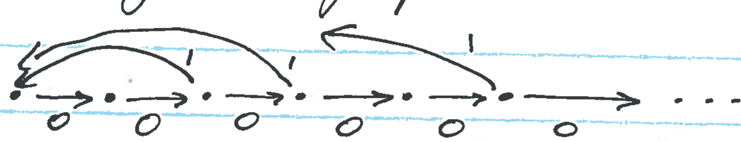
Q

I. Can specification be similarly extended?

II. ______ used to get EDC + CLT?

Toy example: S-gap shifts

- Fix $S \subset \mathbb{N}$, define $X \subset \{0,1\}^{\mathbb{N}}$ by allowing $10^n1$ iff $n \in S$.
- SFT $\Leftrightarrow$ S finite
  sofic $\Leftrightarrow$ S eventually periodic
  specification $\Leftrightarrow$ S non-lacunary (bdd gaps) (syndetic)
- Represent by ctbl graph:




returns only at elements of S.

$\boxed{\text{Rmk}}$ $\{0^n1 \mid n \in S\}$ gives a tower

Say $\mathcal{D} \subset \mathcal{L}$ has spec if satisfies (*) above:

$\boxed{\text{I}}$ $\exists\, \tau\ \forall\, u, v \in \mathcal{D}\ \exists\, w \in \mathcal{L}$ s.t. $|w| \le \tau$ & $uwv \in \mathcal{D}$.

eg. $\mathcal{G} = \{0^n 1 \mid n \in S\}$ has $\boxed{I_0}$ : $\boxed{I}$ w/ $\tau = 0$.

Need $\mathcal{G}$ to be "large enough".

$\boxed{II}$ $\exists\, C^p, C^s \subset \mathcal{L}$ s.t. $\mathcal{L} \subset C^p \mathcal{G} C^s$ and
$h(C^p \cup C^s) < h(\mathcal{L}) = h_{top}(X, f)$

Here $h(\mathcal{D}) = \overline{\lim}_n \frac{1}{n} \log \# \mathcal{D}_n$

-(can generalise everything to $\varphi \neq 0$, $P(\varphi)$)

Write $\mathcal{G}^M = \{uvw \mid u \in C^p, v \in \mathcal{G}, w \in C^s, |u|, |w| \leq M\}$

$\boxed{\text{Thm}}$ (C.-Thompson 2012)

If $\boxed{I}$ holds $\forall\, \mathcal{G}^M$ & $\boxed{II}$ holds for $\mathcal{G}$, then
$(X, \sigma)$ has unique MME. (ES w/ appropriate mods)

$\boxed{Q}$ Does this unique MME have EDC, CLT?
What if $\mathcal{L}$ has spec?

Observation: If $\boxed{I_0}$ + $\boxed{II}$, then tower with
exponential tails $\therefore$ EDC + CLT.

$\boxed{\text{Rmk}}$ Spec seems easier to check in smooth setting
than a tower does. Also $\boxed{I}$ ($\boxed{II}$) go to
factors, while towers do not.

A. Bertrand, 1988: If $X$ has spec then $\exists\ s \in \mathcal{L}$ s.t. $vs, sw \in \mathcal{L} \Rightarrow vsw \in \mathcal{L}$ (synchronising)

- non-symbolic interpretation: expansive + spec $\Rightarrow$ local product structure at some point.

Let $\mathcal{B} = \{v \mid svs \in \mathcal{L}\}$ (bridging words)
& $\mathcal{F} = s\mathcal{B} = \{sv \mid v \in \mathcal{B}\}$
Then $\mathcal{F}$ has $\boxed{I_0}$ - free concatenations ($\therefore$ tower + EDC + CLT)

To address non-uniform case need small extra condition
$\boxed{III}$ Either of the following holds
(a) $\exists\ v \in \mathcal{L}$ s.t. $v\mathcal{L} \cap \mathcal{L} \subset \mathcal{G}\mathcal{C}^s$ (or $\mathcal{L}v \cap \mathcal{L} \subset \mathcal{C}^p\mathcal{G}$)
(b) $w_{[1,k]}, w_{[\ell,|w|]} \in \mathcal{G}$, $\ell \leq k \Rightarrow w \in \mathcal{G}$

Both are natural for some smooth examples:
(a) PH systems (C-Fisher-Thompson)
(b) effective hyperbolicity (C.-Pesin)

$\boxed{\text{Thm}}$ (C., 2014)
- If $\mathcal{G}$ has $\boxed{I}$ then $\exists\ r \in \mathcal{G}$, $c \in \mathcal{L}$ s.t. $s = rcr \in \mathcal{G}$, $\mathcal{B} := \{w \in \mathcal{L} \mid rwr \in \mathcal{G}\}$, $\mathcal{F} := s\mathcal{B} \Rightarrow \mathcal{F}$ has $\boxed{I_0}$
- If $\mathcal{G}$ has $\boxed{II}$ & $\boxed{III}$, so does $\mathcal{F}$: $\exists\ \mathcal{E}^{p,s} \subset \mathcal{L}$ s.t. $\mathcal{L} \subset \mathcal{E}^p\mathcal{F}\mathcal{E}^s$ & $h(\mathcal{E}^p \cup \mathcal{E}^s) < h(X)$.

$\boxed{\text{Cor}}$ $\boxed{I}$ – $\boxed{III} \Rightarrow \exists!\ \mu$ MME, $(X,\mu)$ has tower with exp tails, $\mu$ has EDC & CLT