Bowen-Margulis measure for geodesic flows

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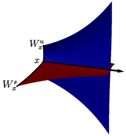
Margulis measure for Anosov flows

Continuous flow on compact metric space: $h_{\text{top}}(F) = \sup_\mu h_\mu(F)$.
Can we describe the measure(s) of maximal entropy?

X a compact manifold, $F = \{f_t\}_{t\in\mathbb{R}}$ Anosov

$T_xX = E^u_x \oplus E^0_x \oplus E^s_x$ integrates to W^u, W^0, W^s



Theorem (Margulis 1970)

There exist "conformal" measures $m^{s,u}_x$ on leaves $W^{s,u}_x$ s.t.

- $(f_t)_* m^s_x = e^h m^s_{f_t x}$ *and* $(f_t)_* m^u_x = e^{-h} m^u_{f_t x}$ *where* $h = h_{\text{top}}(F)$

Product measure $m = m^u_x \times \text{Leb} \times m^s_x$ *is invariant and* $h_m(F) = h$.

- *With* $\text{Per}(T) = \{\text{periodic orbits} : \text{length} \leq T\}$, *the probability measures equidistributed on* $\text{Per}(T)$ *converge to* m *as* $T \to \infty$
- m *is mixing if* F *is topologically mixing, and in this case* $\#\text{Per}(T) \sim \frac{e^{hT}}{hT}$ *(ie.,* $\#\text{Per}(T) \cdot hTe^{-hT} \to 1$ *as* $T \to \infty$*)*

Uniqueness and the Gibbs property

m satisfies the following Gibbs property for all $(x, T) \in X \times \mathbb{R}_+$:

$$C_\epsilon^{-1} e^{-hT} \leq m(\underbrace{\{y : d(f_t y, f_t x) < \epsilon \text{ for all } t \in [0, T]\}}_{\text{Bowen ball } B_T(x,\epsilon)}) \leq C_\epsilon e^{-hT}$$

Theorem (Bowen 1972, 1973)

Given a transitive Axiom A flow, let μ_T be the probability measure equidistributed on $\mathrm{Per}(T)$. Then using the specification property

1. *the limit $m := \lim_{T\to\infty} \mu_T$ exists;*
2. *m has the Gibbs property;*
3. *m is the unique measure of maximal entropy.*

Obtain four ways to characterize the Bowen-Margulis measure m:

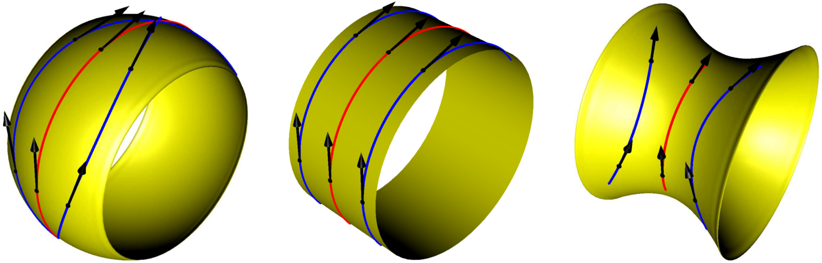
- unique MME
- limit of periodic orbits
- unique Gibbs measure
- conformal leaf measures

Geodesic flow and curvature

M a closed Riemannian manifold, $f_t \colon T^1M \to T^1M$ geodesic flow

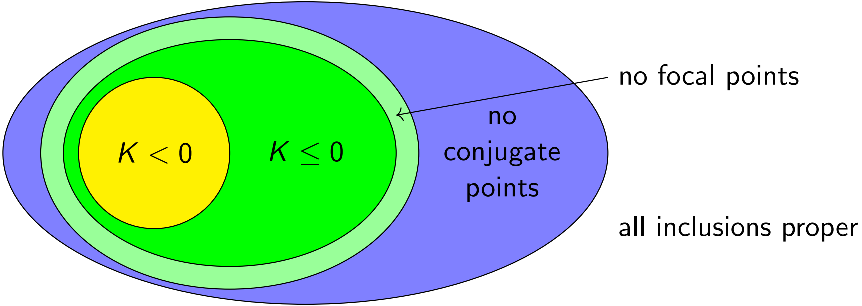
$$v \in T^1M \rightsquigarrow c_v \text{ geodesic with } \dot{c}_v(0) = v \rightsquigarrow f_t(v) := \dot{c}_v(t)$$

Hyperbolicity associated to curvature: $K < 0 \Rightarrow$ Anosov



$K > 0$ $K = 0$ $K < 0$

Hierarchy of hyperbolicity conditions for geodesic flows



No focal points (NFP): balls in universal cover $\widetilde{M}$ are convex

No conjugate points (NCP): $p \neq q \in \widetilde{M}$ determine unique geodesic

Overview of results, and of the talk

Describe leaf measures and specification in old and new settings

Negative curvature (Section 2 of talk)

- Anosov flow, both approaches well-known, including equilibrium states for all Hölder potentials

Nonpositive curvature (Section 3 of talk)

- Leaf measure approach: Knieper (1998), only MME
- Specification approach: Burns–C.–Fisher–Thompson (2018)

No focal points

- Both approaches, with restrictions on φ and/or dimension

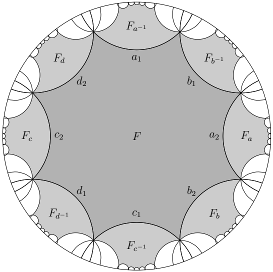
No conjugate points (Section 4 of talk)

- Both approaches done, but only for MME in dimension 2: C.–Knieper–War (2019)

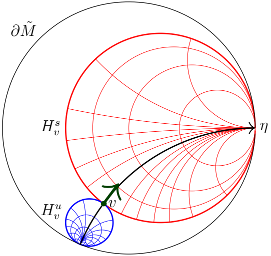
Uniform hyperbolicity and horospheres

M compact, connected, negative curvature $\Rightarrow$ geodesic flow $f_t \colon T^1M \to T^1M$ is topologically mixing and Anosov

1. Go to universal cover $\tilde{M}$



2. Get $E^{s,u}$, $W^{s,u}$ from horospheres



Identify $W_v^{s,u}$ with $H_v^{s,u}$ and hence with ideal boundary $\partial\tilde{M}$

- project $H_v^s \to \partial\tilde{M}$ from $\eta = c_v(\infty)$, and H_v^u from $c_v(-\infty)$

Constructing conformal measures: a rough idea

MME/Gibbs: "every orbit segment of length t gets weight e^{-ht}."

Of course this is nonsense: uncountable! Options to resolve:

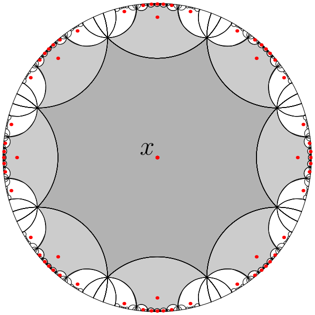
1. Use periodic orbits or (t, ϵ)-separated sets (Bowen)
2. Use isometric action of $\Gamma = \pi_1(M)$ on $\widetilde{M}$ (Patterson–Sullivan)
   Geodesic segment corresponds to pair of points in $\widetilde{M}$

Now with a countable set, can sum:

1. $\sum_{c \text{ periodic orbit}} e^{-h \cdot \text{length}(c)} \text{Leb}_c$
2. $\sum_{\gamma \in \Gamma} e^{-hd(x,\gamma x)} \delta_{\gamma x}$ for $x \in \widetilde{M}$

But these are infinite! Two options:

1. Finite part of sum, normalize, limit
2. Replace h with $s > h$, normalize, take $s \searrow h$



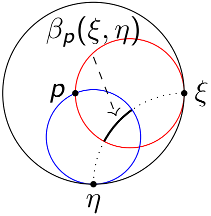
Patterson–Sullivan construction and Margulis measure

Fix reference point $x \in \widetilde{M}$. For each $p \in \widetilde{M}$ get conformal density

$$\nu_p = \lim_{s \searrow h} \left[\text{normalize} \left(\sum_{\gamma \in \Gamma} e^{-sd(p,\gamma x)} \delta_{\gamma x} \right) \right] \qquad (\operatorname{supp} \nu_p = \partial \tilde{M})$$

Can construct a Γ-invariant probability measure $\bar{\mu}$ on $(\partial \tilde{M})^2$ (a geodesic current) by

$$d\bar{\mu}(\xi, \eta) = e^{h\beta_p(\xi,\eta)} \, d\nu_p(\xi) \, d\nu_p(\eta)$$



Using $((\partial \tilde{M})^2 \backslash \text{diag}) \times \mathbb{R} \leftrightarrow T^1\widetilde{M}$, get flow- and Γ-invariant measure on $T^1\widetilde{M}$ via $\bar{\mu} \times \text{Leb}$.

Descend to finite inv. measure on $T^1 M$; normalize to MME μ.

Scaling properties of ν_p w.r.t. p lead to Margulis relations

More about leaf measures

Patterson–Sullivan construction extends to $\varphi \neq 0$, noncompact M, see Paulin, Pollicott, Schapira (2015)

Coding by suspension of Markov shift (Ratner, Bowen, Walters, Ruelle, Series) gives leaf measures via eigendata of Ruelle transfer operator, see Haydn (1994)

Hamenstädt (1989): Margulis leaf measure is Hausdorff measure for an appropriate leaf metric. For $\varphi \neq 0$ make a time change (1997); requires continuous time.

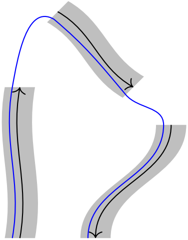
Alternate version: Bowen's definition of h_{top} for noncompact sets (1973) mirrors Hausdorff dimension and gives an outer measure. Restricted to W^u_x this is Margulis leaf measure; see C., Pesin, Zelerowicz (2019) for discrete-time case. Using Pesin–Pitskel' version of pressure (1984), get a corresponding result for $\varphi \neq 0$.

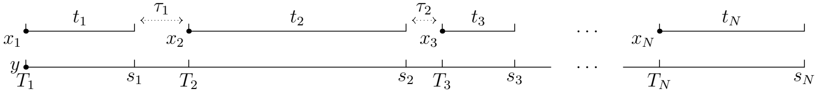
Specification property

Consider a continuous flow f_t on a compact metric space X

Identify $(x,t) \in X \times \mathbb{R}_+$ with orbit segment $\{f_s x : s \in [0,t]\}$

Transitive Anosov $\Rightarrow$ **specification property**:
$\forall$ shadowing scale $\epsilon > 0$ $\exists$ gap size $\tau > 0$ s.t.
$\forall$ list of orbit seg. $\{(x_i,t_i)\}_{i=1}^N \subset X \times \mathbb{R}_+$
$\exists$ ϵ-shadowing τ-connecting orbit:
$y \in X$, $T_j \in \mathbb{R}$ s.t. $f_{T_j}(y) \in B_{t_j}(x_j, \epsilon)$
and $T_{j+1} - (T_j + t_j) \in [0,\tau]$ (see below)





Expansivity + specification $\Rightarrow$ unique MME

Anosov flows are expansive: $\exists \epsilon > 0$ s.t. "bi-infinite Bowen ball" $\Gamma_\epsilon(x) = \{y : d(f_t y, f_t x) \leq \epsilon \ \forall t \in \mathbb{R}\}$ contained in orbit of x.

Theorem (Bowen 1974/75, Franco 1977)

Let f_t be an **expansive** *flow on a compact metric space with the* **specification** *property. Then there is a unique MME μ.*

If f_t has the **periodic** *specification property, then periodic orbits equidistribute to μ.*

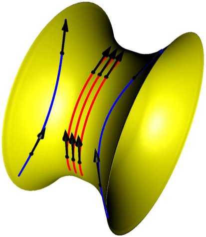
In particular, applies to geodesic flows in negative curvature.

Part of proof is to show $C^{-1} \leq \frac{T \# \operatorname{Per}(T)}{e^{T h_{\text{top}}}} \leq C$.

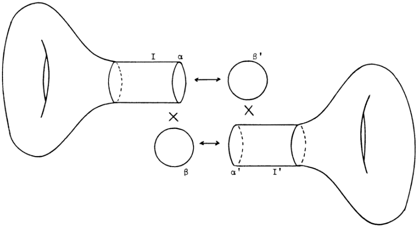
Margulis asymptotics: ratio converges to $1/h_{\text{top}}$ as $T \to \infty$.

Nonpositive curvature: two important examples

> Now suppose M has nonpositive curvature; some sectional curvatures may vanish, but can never be positive.



Example 1: take surface of negative curvature, flatten near a periodic orbit



[Picture: Ballmann, Brin, Eberlein]

Dim > 2: Other possibilities

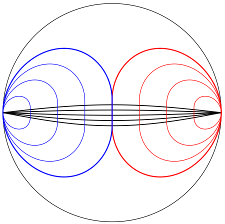
Gromov's example: 3-dim

Some sectional curvature = 0 at every point

No neg. curved metric

Partition into singular (non-hyp) and regular (hyp) parts

Still have universal cover, horospheres, $E^{s,u}$, ...
but now M can have singular geodesics with the following (equivalent) properties:



1. $\exists$ non-trivial parallel Jacobi field
2. Horospheres have higher-order tangency
3. $E^{s,u}$ no longer transverse

$$\mathrm{Sing} = \{v \in T^1M : c_v \text{ is singular}\} \qquad \mathrm{Reg} = T^1M \setminus \mathrm{Sing}$$

$\mu \in \mathcal{M}_f$ is hyperbolic (all Lyapunov exp. $\neq 0$) iff $\mu(\mathrm{Reg}) = 1$

M is rank 1 if $\mathrm{Reg} \neq \emptyset$; then Reg is open, dense, and invariant

- Example 1: Sing is a union of (possibly degenerate) flat strips
- Gromov's example: central strip + all orbits staying in one half

Patterson–Sullivan–Knieper

Theorem (Knieper 1998)

If M has rank 1, then it has a unique MME μ, which is given by a Patterson–Sullivan construction. The MME μ is fully supported and is the limiting distribution of periodic orbits.

Guarantees entropy gap $h_{\mathrm{top}}(\mathrm{Sing}) < h_{\mathrm{top}}(T^1M)$.

- Automatic in dim 2. In higher dimensions gap can be small; modify Gromov's example to have arbitrarily long 'neck'

Patterson–Sullivan product structure gives mixing (Babillot 2002)

Countable Markov partitions give Bernoulli when $\dim M = 2$ (Ledrappier, Lima, Sarig 2016).

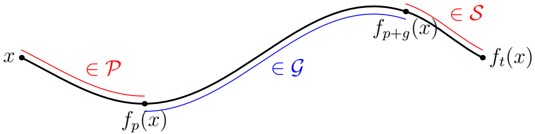
Specification approach on next slides gives Bernoulli in any dimension (Call, Thompson 2019); idea from Ledrappier (1977)

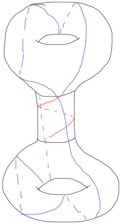
Decompositions of the space of orbit segments

f_t a flow on a compact metric space X. A subset $\mathcal{G} \subset X \times \mathbb{R}_+$ represents a collection of finite-length orbit segments.

> $\mathcal{G}$ has specification if $\forall \epsilon > 0\ \exists \tau$ s.t. every list of orbit segments $\{(x_i, t_i)\}_{i=1}^k \subset \mathcal{G}$ has an ϵ-shadowing τ-connecting orbit.
>
> *Same idea as before, but only needed for* ***good*** *orbit segments*

Decomposition: $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times \mathbb{R}_+$ and functions $p, g, s \colon X \times \mathbb{R}_+ \to \mathbb{R}_+$ s.t. $(p + g + s)(x, t) = t$ and $(x, p) \in \mathcal{P}$, $(f_p x, g) \in \mathcal{G}$, $(f_{p+g} x, s) \in \mathcal{S}$.





Idea: $\mathcal{P}, \mathcal{S}$ are "obstructions to specification"; can glue **if** we first remove pre-/suffixes from $\mathcal{P}, \mathcal{S}$. Need obstructions to be "small"

Decomposition for geodesic flow

Given $v \in T^1M$, let $H^s(v)$ be stable horosphere, $\mathcal{U}^s(v)$ its second fundamental form, and $\lambda^s(v) \geq 0$ the smallest eigenvalue of $\mathcal{U}^s(v)$. Similarly for $\lambda^u(v) \geq 0$, and then $\lambda = \min(\lambda^s, \lambda^u)$.

- $\lambda \colon T^1M \to \mathbb{R}_+$ is a lower bound for curvature of horospheres, and thus bounds contraction/expansion rates

Fix $\eta > 0$ and let $\mathcal{P} = \mathcal{S} = \mathcal{B} = \{(v, T) : \lambda(f_t v) \leq \eta \ \forall t \in [0, T]\}$. Removing longest such prefix and suffix, remaining part is in $\mathcal{G} = \{(v, T) : \lambda(v) \geq \eta \text{ and } \lambda(f_T v) \geq \eta\}$.

- $\mathcal{G}$ has specification (for each fixed $\eta > 0$)
  use transitivity + local product structure on regular set
- This works for the MME, but for $\varphi \neq 0$ need to be more careful: then $\mathcal{B}$ should include all orbit segments with small *average* λ, and $\mathcal{G}$ will be related to *hyperbolic times* (Alves)

Small obstructions $\Rightarrow$ uniqueness

Entropy of obstructions to specification:

- $Q_n = \{x \in X : (x,t) \in \mathcal{P} \cup \mathcal{S} \text{ for some } t \in [n, n+1]\}$
- $\Lambda_n([\mathcal{P} \cup \mathcal{S}], \epsilon) := \max \#\{(n,\epsilon)\text{-separated } E \subset Q_n\}$
- $h([\mathcal{P} \cup \mathcal{S}]) = \lim_{\epsilon \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n} \log \Lambda_n([\mathcal{P} \cup \mathcal{S}], \epsilon)$

Entropy of obstructions to expansivity:

- $\Gamma_\epsilon(x) = \{y \in X : d(f_t y, f_t x) \le \epsilon \ \forall t \in \mathbb{R}\}$
- $\mathrm{NE}(\epsilon) = \{x \in X : \Gamma_\epsilon(x) \not\subset f_{[-s,s]}(x) \text{ for any } s > 0\}$
- $h^\perp_{\mathrm{exp}} = \lim_{\epsilon \to 0} \sup\{h_\mu(f) : \mu(\mathrm{NE}(\epsilon)) = 1\}$

Theorem (V.C., Dan Thompson 2016)

Suppose $h^\perp_{\mathrm{exp}} < h_{\mathrm{top}}$ and $\exists$ decomposition $\mathcal{P}, \mathcal{G}, \mathcal{S}$ s.t. $\mathcal{G}$ has specification and $h([\mathcal{P} \cup \mathcal{S}]) < h_{\mathrm{top}}$. Then $\exists$ a unique MME.

Applying the general result: Sing controls obstructions

Theorem (Keith Burns, V.C., Todd Fisher, Dan Thompson 2018)

For any rank 1 geodesic flow, obstructions to specification and expansivity have entropy $< h_{\text{top}}$*, and thus there is a unique MME.*

Covers many $\varphi \neq 0$, which was the original motivation.

Step 1: direct proof that $h_{\text{top}}(\text{Sing}) < h_{\text{top}}(T^1M)$

Step 2: $\mathrm{NE}(\epsilon) \subset \text{Sing}$, so $h^{\perp}_{\exp} \leq h_{\text{top}}(\text{Sing})$

Step 3: $\lim_{\eta \to 0} h([\mathcal{B}]) = h_{\text{top}}(\text{Sing})$

$\mathcal{M}(\mathcal{B}) \subset \mathcal{M}_\eta = \{\mu : \int \lambda \leq \eta\} \Rightarrow \bigcap_\eta \mathcal{M}(\mathcal{B}) \subset \bigcap \mathcal{M}_\eta = \mathcal{M}_0$

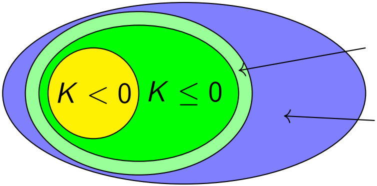
Knieper gets uniqueness *first*, and entropy gap as a corollary.

Extensions to no focal points

No focal points:

- Fei Liu, Fang Wang, Weisheng Wu (arXiv 2018), any dim
  - Patterson–Sullivan approach (following Knieper)
- Katrin Gelfert, Rafael Ruggiero (2019), dimension 2
  - MME via semi-conjugacy to expansive flow with specification
- Dong Chen, Nyima Kao, Kiho Park (arXiv 2018), dimension 2
  - following BCFT (nonuniform specification), get some $\varphi \neq 0$

Comparison of curvature conditions



no focal points: balls in $\widetilde{M}$ convex

no conjugate points: $p \neq q \in \widetilde{M}$
determine unique geodesic

things in $\widetilde{M}$	$K < 0$	$K \leq 0$	NFP	NCP
$t \mapsto d(c_1(t), c_2(t))$ when $c_1(0) = c_2(0)$	strictly convex	convex	monotonic	positive
horospheres	str. cvx	convex		???
$v \mapsto E^{s,u}_v = T_v W^{s,u}_v$	Hölder	continuous		???
$c_1(\pm\infty) = c_2(\pm\infty)$	$c_1 = c_2$	flat strip		???

Available tools in NCP

Theorem (V.C., Gerhard Knieper, Khadim War – arXiv 2019)

Let M be a 2-dimensional Riemannian manifold of genus ≥ 2 with no conjugate points. Then there is a unique MME.

Step 1: $\widetilde{M}$ a disc; unique geodesic $\forall p, q$; $\partial \tilde{M}$ and $H_v^{s,u}$ still exist

Step 2: $h_{\exp}^{\perp}(\epsilon) = 0$ for all $\epsilon \leq \frac{1}{3}$ inj M: $h_\mu > 0 \Rightarrow \mu(\mathrm{NE}(\epsilon)) = 0$

- $w \in \Gamma_\epsilon(v) \Rightarrow \tilde{w} \in \Gamma_\epsilon(\tilde{v})$, so $v \in \mathrm{NE}(\epsilon) \Rightarrow H_v^s \cap H_v^u$ nontrivial
- $h_\mu > 0 \Rightarrow \mu$-a.e. v has $W_v^s \cap W_v^u$ trivial $\therefore$ $H_v^s \cap H_v^u$ trivial

Step 3: There is a *different* metric g_0 with negative curvature...

Morse Lemma: Let g, g_0 be two metrics on M s.t. g_0 has negative curvature and g has no conjugate points. Then there is $R > 0$ such that $\forall p, q \in \widetilde{M}$, the g-geodesic and g_0-geodesic connecting p to q have Hausdorff distance $\leq R$.

Coarse specification via the Morse lemma

For NCP, bijection between $T^1M \times (0,\infty)$ and $(\widetilde{M}^2 - \text{diag})/\pi_1 M$

$$\begin{tikzcd}[column sep=small]
& (\widetilde{M}^2 - \text{diag})/\pi^1 M \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\begin{array}{c} T^1M \times (0,\infty) \\ g\text{-orbit segments} \end{array} \arrow[rr, leftrightarrow, "R\text{-shadowing}"] & & \begin{array}{c} T^1M \times (0,\infty) \\ g_0\text{-orbit segments} \end{array}
\end{tikzcd}$$

Now given orbit segments $(x_1, t_1), \dots, (x_k, t_k)$ for g,

- R-shadow each one by an orbit segment for g_0;
- R-shadow this list by a single g_0 orbit segment (g_0-spec.);
- R-shadow this single orbit segment by a g-orbit segment.

Thus the g-geodesic flow has specification at scale $(\approx)\ 3R$

A uniqueness result at finite scale

Theorem (V.C., Dan Thompson 2016)

X compact metric space, $f_t\colon X \to X$ continuous flow, $\epsilon > 40\delta > 0$.

Assume: $h^{\perp}_{\exp}(\epsilon) < h_{\mathrm{top}}(f)$. $\sup\{h_\mu : \Gamma_\epsilon(x) \not\subset f_{[-t,t]}(x)\ \mu\text{-a.e.}\}$

Assume: Flow has specification at scale δ.

Then $(X, \{f_t\})$ has a unique measure of maximal entropy.

Surface M of genus ≥ 2 with no conjugate points:

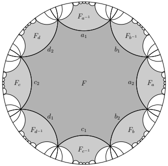
- the geodesic flow has $h^{\perp}_{\exp}(\frac{1}{3}\operatorname{inj} M) = 0 < h_{\mathrm{top}}$;
- the flow has specification at scale $3R$. (R from Morse)

If $40 \cdot 3R < \frac{1}{3}\operatorname{inj} M$, then the general theorem gives a unique MME.

But we have no reason to expect this... probably R is very large.

Salvation by residual finiteness

Solution: Replace M with a finite cover N with $\operatorname{inj} N > 360R$.



- Entropy-preserving bijection between $\mathcal{M}_f(T^1M)$ and $\mathcal{M}_f(T^1N)$
- Theorem gives unique MME on T^1N
- Thus there is a unique MME on T^1M

Why possible? dim $M = 2$ implies $\pi_1(M)$ is *residually finite*.

Patterson–Sullivan and counting closed geodesics

One can carry out the Patterson–Sullivan construction and prove that it gives an MME μ.

Once uniqueness is proved (via specification), it follows that μ is the unique MME, and hence is ergodic.

With ergodicity in hand, product structure gives mixing via Babillot's argument.

Once mixing is known, we can follow approach of Russell Ricks (arXiv 2019 for CAT(0)) to deduce asymptotic estimates on number of (free homotopy classes of) periodic orbits:

$$\lim_{T\to\infty} \frac{P(T)hT}{e^{hT}} = 1 \text{ where } h = h_{\text{top}}(F).$$

Higher dimensions and open questions

Method works for higher-dim M with no conjugate points **if**

1. $\exists$ Riemannian metric g_0 on M with negative curvature;
2. divergence property: $c_1(0) = c_2(0) \Rightarrow d(c_1(t), c_2(t)) \to \infty$;
3. $\pi_1(M)$ is residually finite;
4. $\exists h^* < h_{\text{top}}$ such that if μ-a.e. v has non-trivially overlapping horospheres, then $h_\mu \leq h^*$.

First is a real topological restriction: rules out Gromov example.

Second and third might be redundant? No example satisfying (1) where they are known to fail

Fourth is true if $\{v : H^s_v \cap H^u_v \text{ trivial}\}$ contains an open set. Unclear if this is always true.

What about $\varphi \neq 0$? Not clear how to extend these techniques.