Math 4377/6308 Advanced Linear Algebra I Spring 2013

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FINAL EXAM

Thursday, May 9, 2013

You must give complete justification for all answers in order to receive full credit.

Name: ______________________________

	Points	Possible
Problem 1		/15
Problem 2		/15
Problem 3		/10
Problem 4		/10
Problem 5		/20
Problem 6		/20
Problem 7		/10
Total		/100

1. (a) Let V be a vector space over a field F, and let $S \subset V$. Define what it means for S to be linearly independent. [*5 points*]

(b) Consider the vector space $\mathbb{P}_3(\mathbb{R})$ consisting of polynomials with degree 3 or less, and the subset

$$S = \{1 + x^3,\ x + x^2,\ x^2 - x^3,\ 1 - x\} \subset \mathbb{P}_3(\mathbb{R}).$$

Does S span $\mathbb{P}_3(\mathbb{R})$? [*10 points*]

2. For each of the following linear transformations, determine whether it is (i) one-to-one, (ii) onto, and/or (iii) an isomorphism.

(a) $T\colon \mathbb{M}_{m\times n}(\mathbb{R}) \to \mathbb{M}_{n\times m}(\mathbb{R})$ given by $T(A) = A^t$. [*5 points*]

(b) $T\colon \mathbb{P}(\mathbb{R}) \to \mathbb{P}(\mathbb{R})$ given by $(T(f))(x) = \int_0^x f(t)\,dt$. [*5 points*]

(c) $T\colon M_{n\times n}(F) \to F$ given by $T(A) = \mathrm{Tr}(A)$. [*5 points*]

3. (a) Consider the linear transformation $T\colon \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ given by $(Tf)(x) = (x-1)f'(x) + f''(x)$. Find the matrix $[T]_\beta$, where $\beta = \{1, x, x^2, x^3\}$ is the standard basis for $\mathbb{P}_3(\mathbb{R})$. [*5 points*]

(b) Let $A = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. Compute $[L_A]_\beta$, where $L_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is left multiplication by A. [*5 points*]

4. Let V be a vector space and $T\colon V \to V$ a linear transformation. Prove that $T^2 = 0$ if and only if $R(T) \subset N(T)$. [*10 points*]

5. (a) Label each of the following statements true if it holds for all matrices $A, B, C \in \mathbb{M}_{n\times n}(F)$, and false if there are matrices A, B, C for which it fails. [*12 points*]

(i) $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$

(ii) $\det(ABC) = \det(CAB)$

(iii) $\mathrm{Tr}(ABC) = \mathrm{Tr}(CBA)$

(iv) $\det(ABC) = \det(CBA)$

(v) $\mathrm{Tr}(A^{-1}) = (\mathrm{Tr}(A))^{-1}$ whenever A is invertible

(vi) $\det(A^{-1}) = (\det(A))^{-1}$ whenever A is invertible

(b) Consider the permutation $\pi = (3\ 4\ 1\ 2)$. Write down the matrix of this permutation (this is a 4×4 matrix of 0s and 1s). Determine whether this permutation is even or odd. [*3 points*]

(c) Compute $\det \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 4 & -1 & 0 & 0 \end{pmatrix}$. [*5 points*]

6. Consider the matrix $A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & 5 \end{pmatrix}$.

(a) Find the eigenvalues of A and their algebraic multiplicities. *(Hint: when you try to factor the cubic polynomial, you will find that it has $(t-1)$ as a factor.)* [*7 points*]

(b) Continue working with the matrix $A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & 5 \end{pmatrix}$.

For each of the eigenvalues you found in part (a), find a basis for the corresponding eigenspace, and determine the geometric multiplicity. [*8 points*]

(c) Find a matrix Q such that $Q^{-1}AQ$ is diagonal. [*5 points*]

7. (a) The matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & \sqrt{3} \end{smallmatrix}\right)$ has an eigenvalue $\lambda = \frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{i\pi/6} \in \mathbb{C}$ with associated complex eigenvector $v = (2, \sqrt{3} + i) \in \mathbb{C}^2$. Find a matrix $Q \in \mathbb{M}_{2\times 2}(\mathbb{R})$ and real numbers $r, \theta \in \mathbb{R}$ such that $Q^{-1}AQ$ has the form $r \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. *[5 points]*

(b) Given $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$, write a formula for $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)^k$. Use induction to prove that this formula holds for all $k \in \mathbb{N}$. *[5 points]*

(c) **Extra credit:** Fix $\lambda \in \mathbb{R}$ and let $J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$.

Find a formula for J^k that works for any $k \in \mathbb{N}$, and use induction to prove that your formula is true.

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