Math 4377/6308 Advanced Linear Algebra I Spring 2013

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MIDTERM TEST #2

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You must give complete justification for all answers in order to receive full credit.

Name: ______________________________

	Points	Possible
Problem 1		/10
Problem 2		/15
Problem 3		/20
Problem 4		/20
Problem 5		/20
Problem 6		/15
Total		/100

1. Let $\{v_1, v_2\}$ be a basis for $\mathbb{R}^2$. Is $\{2v_1 + v_2, 2v_1 - v_2\}$ necessarily a basis for $\mathbb{R}^2$? *[10 points]*

YES. Because $\dim \mathbb{R}^2 = 2$, it is enough to check ~~that~~ $\{2v_1 + v_2, 2v_1 - v_2\}$ is linearly independent.

Suppose $a, b \in \mathbb{R}$ are such ~~that~~

$$a(2v_1 + v_2) + b(2v_1 - v_2) = \vec{0}.$$

Then $(2a + 2b)v_1 + (a - b)v_2 = \vec{0}$, and since $\{v_1, v_2\}$ is linearly independent, $2a + 2b = 0$ & $a - b = 0$.

The second equation gives $a = b$, so ~~the~~ first gives $4a = 0$, ~~thus~~ $a = b = 0$. This shows $\{2v_1 + v_2, 2v_1 - v_2\}$ is linearly ind. $\therefore$ it is a basis because it has $2 = \dim \mathbb{R}^2$ elements.

2. Determine whether or not each of the following functions $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. If it is not linear, prove that it is not linear. If it is linear, write the matrix $[T]_\beta$, where β is the standard ordered basis for $\mathbb{R}^2$.

(a) $T(a_1, a_2) = (\sin(a_1), a_2)$ *[5 points]*

Not linear. $T(\frac{\pi}{2}, 0) = (1, 0)$, but

$T(\pi, 0) = (0, 0) \neq 2T(\frac{\pi}{2}, 0)$.

(b) $T(a_1, a_2) = (a_1, a_2 + 1)$ *[5 points]*

Not linear. $T(0, 0) = (0, 1)$, but every linear ~~transformation~~ sends $\vec{0}$ to $\vec{0}$.

(c) $T(a_1, a_2) = (2a_1 + a_2, 3a_1 - 2a_2)$ *[5 points]*

Linear

$$[T]_\beta = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$$

3. Let V and W be vector spaces and $T\colon V \to W$ a linear transformation. Suppose that T is one-to-one and that S is a subset of V.

(a) State what it means for S to be linearly independent. [*5 points*]

S is linearly dependent if $\exists\ v_1, \dots, v_n \in S$ & $a_1, \dots, a_n \in F$ s.t. not all the a_j are 0, and $\sum a_j v_j = \vec{0}$.

S is linearly independent if it is not linearly dependent.

(b) Prove that S is linearly independent if and only if $T(S)$ is linearly independent. [*15 points*]

Suppose S is linearly dependent. Then $\exists\ v_1, \dots, v_n \in S$ & $a_1, \dots, a_n \in F$, not all 0, such that $\sum a_j v_j = \vec{0}$. By linearity we have $\sum a_j T(v_j) = \vec{0}$ & because T is 1-1 the vectors $T(v_j)$ are distinct, so $T(S)$ is linearly dependent.

Now suppose $T(S)$ is linearly dependent. Then $\exists\ v_1, \dots, v_n \in S$ & $a_1, \dots, a_n \in F$, not all 0, such that $\sum a_j T(v_j) = \vec{0}$. By linearity, $T(\sum a_j v_j) = \vec{0}$, and because T is 1-1, this implies $\sum a_j v_j = \vec{0}$. Thus S is linearly dependent.

This shows S dependent $\iff$ $T(S)$ dependent, which is equivalent to S independent $\iff$ $T(S)$ independent.

4. Consider the ordered bases β and β' for $\mathbb{R}^2$ given by

$$\beta = \{(1,1), (1,-1)\}, \qquad \beta' = \{(3,1), (-1,3)\}.$$

(a) Find the change of coordinate matrix Q that changes β'-coordinates into β-coordinates. *[10 points]*

$$v_1 = \begin{pmatrix}1\\1\end{pmatrix} \quad v_2 = \begin{pmatrix}1\\-1\end{pmatrix} \quad w_1 = \begin{pmatrix}3\\1\end{pmatrix} \quad w_2 = \begin{pmatrix}-1\\3\end{pmatrix}$$

$$w_1 = 2v_1 + v_2 \Rightarrow [w_1]_\beta = \begin{bmatrix}2\\1\end{bmatrix}$$

$$w_2 = v_1 - 2v_2 \Rightarrow [w_2]_\beta = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$\left.\begin{aligned} Qe_1 &= Q[w_1]_{\beta'} = [w_1]_\beta = \begin{bmatrix}2\\1\end{bmatrix} \\ Qe_2 &= Q[w_2]_{\beta'} = [w_2]_\beta = \begin{bmatrix}1\\-2\end{bmatrix}\end{aligned}\right\} \Rightarrow Q = \begin{bmatrix}2 & 1\\1 & -2\end{bmatrix}$$

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and let $A = [T]_\beta$. Write an expression for $[T]_{\beta'}$. *[10 points]*

$$[T]_{\beta'} = \underset{\beta'\leftarrow\beta}{Q}\,[T]_\beta\,\underset{\beta\leftarrow\beta'}{Q}$$

$$= \boxed{Q^{-1}AQ}$$

5. For each of the following linear transformations, determine whether it is (i) one-to-one, (ii) onto, and/or (iii) an isomorphism.

(a) $T\colon \mathbb{M}_{m\times n}(\mathbb{R}) \to \mathbb{M}_{n\times m}(\mathbb{R})$ given by $T(A) = A^t$. [*10 points*]

1-1: Yes, because $A^t = 0 \Rightarrow A = (A^t)^t = 0^t = 0$.

onto: Yes, because $A = (A^t)^t$

isomorphism: Yes, because 1-1 & onto.

(b) $T\colon \mathbb{P}(\mathbb{R}) \to \mathbb{P}(\mathbb{R})$ given by $(T(f))(x) = f'(x)$. [*10 points*]

1-1: No, because $f(x) \equiv c$ (a constant) implies $f'(x) = 0$ $\forall\ c \in \mathbb{R}$, so kernel is non-trivial

onto: Yes, because $f = Tg$ if $g \in \mathbb{P}(\mathbb{R})$ is defined by $g(x) = \int_0^x f(y)\,dy$

isomorphism: No, because not 1-1.

6. (a) Give a matrix A such that $A \neq 0$ but $A^2 = 0$. [*5 points*]

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) Give matrices A, B such that $AB = I$ but $BA \neq I$. [*5 points*]

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Consider the matrix

$$A = \begin{pmatrix} 5 & 0 & -1 & 2 \\ 2 & 1 & 0 & 3 \\ -1 & 1 & 1 & -2 \\ -3 & 0 & 3 & 4 \end{pmatrix}.$$

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the standard (column) basis vectors in $\mathbb{R}^4$. Compute $A\mathbf{e}_1$, $A\mathbf{e}_2$, $A\mathbf{e}_3$, and $A\mathbf{e}_4$. [*5 points*]

$$Ae_1 = \begin{bmatrix} 5 \\ 2 \\ -1 \\ -3 \end{bmatrix} \quad Ae_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$Ae_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \quad Ae_4 = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 4 \end{bmatrix}$$

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