Math 4377/6308	Advanced Linear Algebra I	Fall 2013
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Final Exam Review

You are responsible for all material covered in lectures, assignments, and tests. As with the two tests earlier in the term, you are responsible for knowing the definitions of all the terms below. Also listed are some common procedures you should know how to do and some general results you should know. The exam will be similar in structure to the tests, and will include definitions, short proofs, and calculations.

Vector spaces.

- Examples: $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{P}_n$, $\mathbb{P}$, $C(\mathbb{R})$, $C^1(\mathbb{R})$.
- Determining whether something is or is not a vector space

Subspaces.

- Examples, determining whether something is a subspace.
- Sums of subspaces.
- Direct sums, determining whether or not a given sum is direct.
- Existence of a complement of a subspace W in finite-dimensional spaces (another subspace X such that $V = W \oplus X$) – Theorem 4.11

Linear combinations and span.

- Determining whether or not a vector is in the span of a set.
- Determining whether or not a set spans the entire vector space.
- Relationship to non-homogeneous systems of linear equations.

Linear dependence and independence.

- Determining whether or not a given set is linearly dependent.
- If S is spanning and L is linearly independent, then $\#L \leq \#S$ – Proposition 3.4
- Behaviour of linearly dependent and independent sets under adding or removing vectors – Exercise 3.3.
- S is linearly dependent iff there is $v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$, iff there is $v \in S$ such that $\operatorname{span}(S \setminus \{v\}) = \operatorname{span}(S)$ – Proposition 3.5
- S is linearly independent iff everything in $\operatorname{span} S$ can be represented in a *unique* way – Proposition 3.6

Bases and dimension

- Every basis has the same number of vectors – Theorem 3.9
- Finite spanning set implies existence of a finite basis – Lemma 4.5
- Every linearly independent set can be completed to a basis if V is finite-dimensional – Theorem 4.7

- Determining whether a given set is a basis. If the number of vectors in B is not equal to $\dim V$, then B is not a basis. If it is equal, then it is only necessary to check either linear independence or spanning, and the other follows.

Quotient spaces

- Equivalence modulo a subspace, congruence classes.
- Congruence classes are affine subspaces: $[v]_Y = v + Y$. (Proposition 5.4)
- Formula for dimension of quotient space – Theorem 5.7

Linear maps

- Examples: differentiation, evaluation, matrix multiplication, rotation.
- Determining whether a map is linear.
- Behaviour of linear dependence and independence: a dependent set gets mapped to a dependent set (Proposition 6.13). An independent set gets mapped to an independent set if the map is 1-1 (Exercise 6.14).
- A spanning set gets mapped to a spanning set if T is onto – Exercise 6.15
- Images and preimages of subspaces are subspaces – Theorem 6.16
- A linear transformation is uniquely determined by its action on a basis – Lemma 7.7

Nullspace and range

- Determining the nullspace and range of a map.
- T is 1-1 iff N_T is trivial – Exercise 7.4
- Solution sets of systems of equations and ODEs as nullspaces.
- Dimension of nullspace + dimension of range = dimension of domain (Theorem 9.1)

Isomorphisms

- If T is an isomorphism, then B is a basis if and only if $T(B)$ is – Prop 7.6
- Two finite-dimensional vector spaces are isomorphic iff they have the same dimension – Theorem 7.8

Compositions of linear maps

- Associativity and distributivity
- Non-commutativity.
- Transpose of a linear transformation, relationship to transpose of a matrix.

Representing linear maps with matrices.

- Finding matrix of linear map relative to standard bases for K^n and K^m (Theorem 10.1)
- Finding matrix of linear map relative to any bases β and γ

Change of coordinates.

- Interpreting commutative diagrams
- Isomorphism $I_\beta \colon K^n \to V$ corresponding to a basis β for V
- Finding change-of-coordinates matrix I_β^γ that transforms β-coordinates to γ-coordinates
- Finding $[v]_\gamma$ using I_β^γ and $[v]_\beta$
- Finding $[T]_\gamma$ using I_β^γ and $[T]_\beta$: conjugate (similar) matrices

Nilpotent operators, projections.

- Definition of nilpotent ($T^k = \mathbf{0}$)
- Two equivalent definitions of projection: one in terms of R_T, N_T, the other in terms of $T^2 = T$.
- Strictly upper triangular matrices are nilpotent

Row and column rank.

- Definition of row space, column space, row rank, column rank.
- Row rank and column rank are equal.

Eigenvectors and eigenvalues.

- Definition of eigenvector and eigenvalue
- Eigenspace for λ is nullspace for $A - \lambda I$
- Finding eigenvectors for a given eigenvalue
- Example of a linear transformation in $\mathbb{R}^2$ with no eigenvalues
- Every linear transformation in $\mathbb{C}^n$ has an eigenvalue (Proposition 14.4)

Determinants

- Properties that characterise the determinant as a function of n vectors in K^n (Lecture 17): multilinear, vanishes if two input vectors are equal, normalised, alternating, vanishes if input vectors are linearly dependent
- Formula for determinant via sums of permutations
- Finding the sign of a permutation via crossing diagrams, backwards pairs, and number of transpositions
- Determinant of transpose ($\det A^t = \det A$)
- Determinant of products $\det(AB) = (\det A)(\det B)$; similar matrices have same determinant
- Invertibility characterised by determinant: invertible iff $\det A \neq 0$
- Formula for determinant via cofactor expansion
- Finding determinant via row reduction
- Formula for inverse of a matrix using determinant. Cramer's rule
- Interpretation of determinant as a signed volume

Trace

- $\text{Tr}(AB) = \text{Tr}(BA)$
- Similar matrices have the same trace

Spectral theory

- Definitions: Characteristic polynomial, spectrum, algebraic and geometric multiplicities, diagonalisable
- Similar matrices have the same spectrum and multiplicities
- Digaonalisable iff exists a basis of eigenvectors (Theorem 21.8)
- Geometric multiplicity is less than or equal to algebraic multiplicity (Theorem 21.11). The two are equal for all eigenvalues iff the matrix is diagonalisable (Corollary 22.2)
- In $\mathbb{C}$, trace is sum of eigenvalues, determinant is product of eigenvalues (Theorem 22.3)
- Spectral mapping theorem (Theorem 22.4)
- Cayley–Hamilton theorem (Theorem 23.1)
- Canonical forms for 2×2 matrices over $\mathbb{C}$ – everything is similar to either a diagonal matrix or a Jordan block. (Theorem 23.2)
- Computing powers of matrices using canonical forms
- Canonical forms for 2×2 real matrices – similar to either a diagonal matrix, a Jordan block, or a scaled rotation

Markov chains

- Definitions: Markov chain, probability vector, stochastic matrix, irreducible, primitive, transition matrix
- How to write down transition matrix given a Markov chain as a graph. Transition matrix is always stochastic.