Math 4377/6308 Advanced Linear Algebra I Fall 2013
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HOMEWORK 10 – solutions

Due *4pm Wednesday, November 20.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Suppose that all n eigenvalues are distinct and have $|\lambda_j| < 1$ for all j. Show that $A^N v \to 0$ as $N \to \infty$ for every $v \in K^n$.

Solution. *[10 points]* Let $v_1, \dots, v_n$ be eigenvectors for the eigenvalues $\lambda_1, \dots, \lambda_n$. Because the eigenvalues are distinct, the eigenvectors $v_1, \dots, v_n$ are linearly independent, hence they form a basis for K^n. Given any $v \in K^n$, there are coefficients $a_1, \dots, a_n \in K$ such that $v = \sum_j a_j v_j$, and thus

$$A^N v = \sum_{j=1}^n a_j (A^N v_j) = \sum_{j=1}^n a_j \lambda_j^N v_j.$$

Because $|\lambda_j| < 1$ for all j, we have $\lambda_j^N \to 0$ as $N \to \infty$, and hence the sum also goes to 0.

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2. We showed in class that $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$ for any $A, B, C \in \mathbb{M}_{n\times n}$. Is it always true that $\mathrm{Tr}(ABC) = \mathrm{Tr}(BAC)$? If so, prove it; if not, find a counterexample.

Solution. *[10 points]* This statement is not always true. There are many, many counterexamples. One possibility is $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so that

$$ABC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which has trace 1, but

$$AC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0},$$

so $BAC = \mathbf{0}$, which has trace 0.

3. Let A be an upper triangular matrix. Show that the eigenvalues of A are precisely the diagonal entries of A, and that the algebraic multiplicity of an eigenvalue is the number of times it appears on the diagonal of A.

Solution. *[10 points]* Because A is upper triangular, so is $tI - A$ for every t; indeed, for every $i > j$ we have $(tI - A)_{ij} = tI_{ij} - A_{ij} = t \cdot 0 - 0$, since I is diagonal and A is upper triangular. We recall that the determinant of an upper triangular matrix is the product of its diagonal entries. Thus

$$p_A(t) = \det(tI - A) = \prod_{j=1}^{n} (tI - A)_{jj} = \prod_{j=1}^{n} (t - A_{jj}).$$

This gives a factorisation of the characteristic polynomial into linear factors, and we see that every entry A_{jj} is a root of the characteristic polynomial. Moreover, the algebraic multiplicity is the number of times it appears as a root in the factorisation above, which is exactly the number of times it appears on the diagonal of A.

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4. Let $n = k + m$, and let $A \in \mathbb{M}_{n\times n}$ have the block form $A = \left(\begin{smallmatrix} X & Y \\ \mathbf{0} & Z \end{smallmatrix}\right)$, where $X \in \mathbb{M}_{k\times k}$, $Y \in \mathbb{M}_{k\times m}$, $Z \in \mathbb{M}_{m\times m}$, and $\mathbf{0}$ is the $m \times k$ zero matrix. Show that $\det A = (\det X)(\det Z)$.

Solution. *[10 points]* This can be proved in a number of different ways: using the formula for determinant as a sum over permutations; using cofactor expansion and induction; using the axiomatic characterisation of determinant as a normalised alternating multilinear functions; or using Gaussian elimination (row reduction). We give the argument via cofactor expansion and induction.

We use induction on n, and show that for every n, the claim in the problem is true for every $k + m = n$ and every X, Y, Z with the dimensions specified. The case $n = 2$ is immediate, because here we must have $k = m = 1$ and thus $X = (\,x\,)$, $Y = (\,y\,)$, $Z = (\,z\,)$, and

$\det A = \det\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) = xz = (\det X)(\det Z)$. For the inductive step, suppose n has the property described. Then let k, m be such that $k + m = n + 1$, and let $A \in \mathbb{M}_{(n+1)\times(n+1)}$ have the block form shown. By cofactor expansion along the first column of A, we have

$$(1) \qquad \det A = \sum_{i=1}^{n} (-1)^{1+i} A_{i1} \det \tilde{A}_{i1} = \sum_{i=1}^{k} (-1)^{1+i} A_{i1} \det \tilde{A}_{i1},$$

where the last inequality uses the fact that $A_{i1} = 0$ for every $i > k$, due to the block form of A. Note that when $1 \leq i \leq k$, the matrix $\tilde{A}_{i1}$ has the block form

$$\tilde{A}_{i1} = \begin{pmatrix} \tilde{X}_{i1} & \tilde{Y}_{i,\emptyset} \\ \mathbf{0} & Z \end{pmatrix}$$

where $\tilde{Y}_{i,\emptyset}$ denotes the $(k-1) \times m$ matrix obtained from Y by removing the ith row (but not removing any columns). By the inductive hypothesis, we have

$$\det \tilde{A}_{i1} = (\det \tilde{X}_{i1})(\det Z),$$

and so (1) gives

$$\det A = \sum_{i=1}^{k} (-1)^{1+i} A_{i1} (\det \tilde{X}_{i1})(\det Z) = (\det X)(\det Z),$$

using the formula for $\det X$ via cofactor expansion along the first column of X. By induction, the result is true for all n.

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5. Fix $\theta \in \mathbb{R}$ such that θ is not a multiple of π, and consider the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that A has no real eigenvalues but has two distinct complex eigenvalues. Find the corresponding complex eigenvectors.

Solution. *[10 points]* The characteristic polynomial of A is

$$p_A(t) = \det \begin{pmatrix} t - \cos\theta & \sin\theta \\ -\sin\theta & t - \cos\theta \end{pmatrix} \qquad = (t - \cos\theta)^2 + \sin^2\theta$$
$$= t^2 - 2(\cos\theta)t + 1,$$

and thus by the quadratic formula, the eigenvalues are

$$\lambda_{1,2} = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta.$$

Because θ is not a multiple of π, we have $\sin\theta \neq 0$, and so these eigenvalues are distinct and non-real. For $\lambda_1 = \cos\theta + i\sin\theta$, we find an eigenvector v_1 by finding the null space of

$$A - \lambda_1 I = \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix}.$$

We see that $v_1 = \left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)$ is an eigenvector for λ_1. Similarly,

$$A - \lambda_2 I = \begin{pmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{pmatrix},$$

and so $v_2 = \left(\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right)$ is an eigenvector for λ_2.

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6. Let $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \in \mathbb{M}_{3\times 3}(\mathbb{R})$.

(i) Determine all the eigenvalues of A.

Solution. *[5 points]* We compute the characteristic polynomial of A: the evaluation of the determinant below is done by summing over permutations:

$$\begin{aligned} p_A(t) &= \det \begin{pmatrix} -t & -2 & -3 \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{pmatrix} \\ &= -t(1-t)(5-t) + (-2)(-1)(2) + (-3)(-1)(2) \\ &\qquad - (-t)(2)(1) - (-1)(-2)(5-t) - (2)(1-t)(-3) \\ &= -t(t^2 - 6t + 5) + 4 + 6 - 2t - 2(5-t) + 6(1-t) \\ &= -t^3 + 6t^2 - 5t + 10 - 2t - 10 + 2t + 6 - 6t \\ &= -t^3 + 6t^2 - 11t + 6 \\ &= -(t^3 - 6t^2 + 11t - 6) \\ &= -(t-1)(t^2 - 5t + 6) \\ &= -(t-1)(t-2)(t-3). \end{aligned}$$

The roots are $1, 2, 3$, so the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

Solution. *[5 points]* For λ_1, we find the null space of $A - \lambda_1 I$ by row reducing

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -2 & -2 \\ -1 & 0 & -1 \\ 0 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so that $(A - \lambda_1 I)\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = 0$ reduces to $x_2 + x_3 = x_1 + x_3 = 0$, and we see that

$$N(A - \lambda_1 I) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

The set of eigenvectors for $\lambda_1 = 1$ is all nonzero multiples of $v_1 = \left(\begin{smallmatrix} 1 \\ 1 \\ -1 \end{smallmatrix}\right)$.

Similarly for λ_2, we row reduce $A - 2I$ as

$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and conclude that the set of eigenvectors for $\lambda_2 = 2$ is all nonzero multiples of $v_2 = \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right)$.

Finally, for λ_3, we row reduce $A - 3I$ as

$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

so the set of eigenvectors for $\lambda_3 = 3$ is all nonzero multiples of $v_3 = \left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right)$.

(iii) Is it possible to find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A? If so, do it, and then determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution. *[10 points]* Yes. The set $\beta = \{v_1, v_2, v_3\}$ found in the previous part is linearly independent, and so forms a basis for $\mathbb{R}^3$. Taking

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix},$$

we see that $D = Q^{-1}AQ$ has

$$D\mathbf{e}_j = Q^{-1}AQ\mathbf{e}_j = Q^{-1}Av_j = Q^{-1}\lambda_j v_j = \lambda_j \mathbf{e}_j,$$

and so

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$