HOMEWORK 10

Due *4pm Wednesday, November 20.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Suppose that all n eigenvalues are distinct and have $|\lambda_j| < 1$ for all j. Show that $A^N v \to 0$ as $N \to \infty$ for every $v \in K^n$.

2. We showed in class that $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$ for any $A, B, C \in \mathbb{M}_{n\times n}$. Is it always true that $\mathrm{Tr}(ABC) = \mathrm{Tr}(BAC)$? If so, prove it; if not, find a counterexample.

3. Let A be an upper triangular matrix. Show that the eigenvalues of A are precisely the diagonal entries of A, and that the algebraic multiplicity of an eigenvalue is the number of times it appears on the diagonal.

4. Let $n = k + m$, and let $A \in \mathbb{M}_{n\times n}$ have the block form $A = \left(\begin{smallmatrix} X & Y \\ \mathbf{0} & Z \end{smallmatrix}\right)$, where $X \in \mathbb{M}_{k\times k}$, $Y \in \mathbb{M}_{k\times m}$, $Z \in \mathbb{M}_{m\times m}$, and $\mathbf{0}$ is the $m \times k$ zero matrix. Show that $\det A = (\det X)(\det Z)$. Is it always true that $\det \left(\begin{smallmatrix} X & Y \\ W & Z \end{smallmatrix}\right) = (\det X)(\det Z) - (\det Y)(\det W)$?

5. Fix $\theta \in \mathbb{R}$ such that θ is not a multiple of π, and consider the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that A has no real eigenvalues but has two distinct complex eigenvalues. Find the corresponding complex eigenvectors.

6. Let $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \in \mathbb{M}_{3\times 3}(\mathbb{R})$.

(i) Determine all the eigenvalues of A.
(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
(iii) Is it possible to find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A? If so, do it, and then determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.