HOMEWORK 11 solutions

Due *4pm Wednesday, December 4.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. For each of the following 2×2 matrices A, find an invertible 2×2 matrix Q with real entries such that $Q^{-1}AQ$ has one of the three real canonical forms: diagonal $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, Jordan block $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$, or scaled rotation $q\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. Use this to compute A^6 by hand.

(a) $A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

Solution. *[10 points]* The characteristic polynomial is

$$t^2 - \operatorname{Tr}(A)t + \det(A) = t^2 - 6t + 9 = (t-3)^2,$$

so A has a single eigenvalue, $\lambda = 3$, with algebraic multiplicity 2. Because A is not a diagonal matrix, the geometric multiplicity of λ is less than the dimension of $\mathbb{C}^2$, thus $m_g(\lambda) = 1$. We have $A - \lambda I = A - 3I = \left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, and so $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ is an eigenvector for A. Recall that a basis of *generalised* eigenvectors can be found by taking $\{v, w\}$, where w is in $N_{(A-3I)^2} = \mathbb{C}^2$ but not $N_{(A-3I)} = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $v = (A-3I)w$. Taking $w = \mathbf{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ gives $v = (A-3I)v_1 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, so we use the matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \qquad Q^{-1}AQ = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Now we have $A = QJQ^{-1}$, so

$$\begin{aligned} A^6 = QJ^6Q^{-1} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3^6 & 6 \cdot 3^5 \\ 0 & 3^6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \\ &= 3^6 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = 3^6 \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \end{aligned}$$

(b) $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$

Solution. *[10 points]* The characteristic polynomial is

$$t^2 - \operatorname{Tr}(A)t + \det(A) = t^2 - 3t + 2 = (t-2)(t-1),$$

so the eigenvalues of A are 2 and 1. The eigenspaces are the null spaces of $\left(\begin{smallmatrix} 1 & 1 \\ -2 & -2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 1 \\ -2 & -1 \end{smallmatrix}\right)$, respectively, so we can take $\beta = \{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right)\}$ as a basis of eigenvectors, which gives

$$Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \qquad Q^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Q^{-1}AQ = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

This lets us compute A^6 as

$$A^6 = QD^6Q^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 127 & 63 \\ -126 & -62 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} 2 & 4 \\ -3 & 2 \end{pmatrix}$

Solution. *[10 points]* Solving for roots of the characteristic polynomial gives $t^2 - 4t + 16 = 0$, so by the quadratic formula we have eigenvalues λ and $\bar{\lambda}$, where $\lambda = 4\left(\frac{1+\sqrt{3}i}{2}\right) = 4e^{\pm i\pi/3}$. We first find a *complex* eigenvector $v \in \mathbb{C}^2$ for $\lambda = 2 + 2\sqrt{3}i$ by observing that the null space of $A - \lambda I = \left(\begin{smallmatrix} -2\sqrt{3}i & 4 \\ -3 & -2\sqrt{3}i \end{smallmatrix}\right)$ is spanned by $v = \left(\begin{smallmatrix} 2 \\ \sqrt{3}i \end{smallmatrix}\right)$. Thus to put the matrix in the canonical form of over $\mathbb{R}$, which will be a scaled rotation, we take $w_1 = \left(\begin{smallmatrix} 0 \\ \sqrt{3} \end{smallmatrix}\right)$ to be the imaginary part of v, and $w_2 = \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)$ to be the real part of v. We get

$$Q = \begin{pmatrix} 0 & 2 \\ \sqrt{3} & 0 \end{pmatrix}, \qquad Q^{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{2} & 0 \end{pmatrix},$$

and conjugating by Q puts A in the form

$$R := Q^{-1}AQ = \begin{pmatrix} 2 & -2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix} = 4\begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix}.$$

Now $R^6 = 4^6\left(\begin{smallmatrix} \cos(2\pi) & -\sin(2\pi) \\ \sin(2\pi) & \cos(2\pi) \end{smallmatrix}\right) = 4^6 I$, and we conclude that

$$A^6 = QR^6Q^{-1} = Q(4^6)IQ^{-1} = 4^6 I = \begin{pmatrix} 4096 & 0 \\ 0 & 4096 \end{pmatrix}.$$

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2. Let λ be an eigenvalue of A. Say that v is a generalised eigenvector of order k if $(A - \lambda I)^k v = \mathbf{0}$ but $w_j := (A - \lambda I)^j v \neq \mathbf{0}$ for $1 \leq j < k$.

(a) Suppose v is a generalised eigenvector of order 2, so that $w_1 = (A - \lambda I)v$ is an eigenvector. Show that for every polynomial $q(t)$ we have $q(A)v = q(\lambda)v + q'(\lambda)w_1$.

Solution. *[10 points]* It was shown in lecture that $A^N v = \lambda^N v + N\lambda^{N-1}w_1$. Recall that this is an easy induction: for $N = 1$ it follows from the fact that $w_1 = (A - \lambda I)v$, while for $N > 1$ the induction step is

$$\begin{aligned} A^{N+1}v &= A(\lambda^N v + N\lambda^{N-1}w_1) \\ &= \lambda^N(\lambda v + w_1) + N\lambda^{N-1}Aw_1 \\ &= \lambda^{N+1}v + (N+1)\lambda^N w_1. \end{aligned}$$

Now given a polynomial $q(t) = \sum_{j=0}^{d} c_j t^j$, we have

$$\begin{aligned} q(A)v &= \left(\sum_{j=0}^{d} c_j A^j\right) v = \sum_{j=0}^{d} c_j (A^j v) \\ &= \sum_{j=0}^{d} c_j(\lambda^j v + j\lambda^{j-1}w_1) \\ &= \left(\sum_{j=0}^{d} c_j \lambda^j\right) v + \left(\sum_{j=0}^{d} j c_j \lambda^{j-1}\right) w_1 \\ &= q(\lambda)v + q'(\lambda)w_1. \end{aligned}$$

(b) Now suppose v is a generalised eigenvector of order 3, so that $w_1 = (A - \lambda I)v$ is a generalised eigenvector of order 2, and $w_2 = (A - \lambda I)w_1$ is an eigenvector. Show that

$$A^N v = \lambda^N v + N\lambda^{N-1}w_1 + \frac{1}{2}N(N-1)\lambda^{N-2}w_2,$$

and use this to derive a formula for $q(A)v$.

Solution. *[10 points]* As before, we have $Av = \lambda v + w_1$, but now instead of $Aw_1 = \lambda w_1$, we have $Aw_1 = \lambda w_1 + w_2$, with $Aw_2 = \lambda w_2$. The formula for $A^N v$ follows immediately when $N = 1$, and the inductive step is

$$\begin{aligned}
A^{N+1}v &= A(\lambda^N v + N\lambda^{N-1}w_1 + \frac{1}{2}N(N-1)\lambda^{N-2}w_2) \\
&= \lambda^N(Av) + N\lambda^{N-1}(Aw_1) + \frac{1}{2}N(N-1)\lambda^{N-2}(Aw_2) \\
&= \lambda^N(\lambda v + w_1) + N\lambda^{N-1}(\lambda w_1 + w_2) + \frac{1}{2}N(N-1)\lambda^{N-2}(\lambda w_2) \\
&= \lambda^{N+1}v + (N+1)\lambda^N w_1 + \frac{1}{2}N(N+1)\lambda^{N-1}w_2.
\end{aligned}$$

Now for $q(t) = \sum_{j=0}^d c_j t^j$ we have

$$\begin{aligned}
q(A)v &= \left(\sum_{j=0}^d c_j A^j\right) v = \sum_{j=0}^d c_j(A^j v) \\
&= \sum_{j=0}^d c_j \left(\lambda^j v + j\lambda^{j-1}w_1 + \frac{1}{2}j(j-1)\lambda^{j-2}w_2\right) \\
&= \left(\sum_{j=0}^d c_j\lambda^j\right) v + \left(\sum_{j=0}^d jc_j\lambda^{j-1}\right) w_1 + \left(\sum_{j=0}^d \frac{1}{2}j(j-1)c_j\lambda^{j-2}\right) w_2 \\
&= q(\lambda)v + q'(\lambda)w_1 + \frac{1}{2}q''(\lambda)w_2.
\end{aligned}$$

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3. Although every real matrix has eigenvalues in $\mathbb{C}$, we have seen examples of 2×2 matrices with no eigenvalues in $\mathbb{R}$. Show that such examples only exist in even dimensions. That is, show that if $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ and n is odd, then A has a real eigenvalue.

Solution. *[10 points]* If A is $n \times n$ with real coefficients and n is odd, then the characteristic $p_A(t)$ is a degree n polynomial with real coefficients. Because n is odd, this polynomial has a real root λ – indeed, as $t \to \infty$ and $t \to -\infty$, the t^n term becomes much larger than all other terms in $p_A(t)$, and since it takes opposite signs for positive and negative t, we see that $p_A(t)$ is sometimes positive and sometimes negative. Thus there must be $\lambda \in \mathbb{R}$ such that $p_A(\lambda) = 0$. This is the desired real eigenvalue.

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4. (a) Suppose $A, B \in \mathbb{M}_{n\times n}(\mathbb{C})$ and B is invertible. Show that there is $c \in \mathbb{C}$ such that $A+cB$ is not invertible. *Hint: look at* $\det(A+cB)$.

Solution. *[5 points]* By multiplicativity of determinant and invertibility of B, we have

$$\det(A + cB) = \det((AB^{-1} + cI)B) = \det(AB^{-1} + cI)\det B.$$

Let $q(t) = p_{AB^{-1}}(t) = \det(AB^{-1} - tI$ be the characteristic polynomial of AB^{-1}, then $\det(A + cB) = q(-c)\det B$. Because every characteristic polynomial has a complex root, there is $\lambda \in \mathbb{C}$ such that $q(\lambda) = 0$. Taking $c = -\lambda$ gives $\det(A+cB) = 0$, hence $A+cB$ is non-invertible.

(b) Give an example of matrices A, B for which $A+cB$ is invertible for every $c \in \mathbb{C}$. *Hint: by the previous part, B must be non-invertible.*

Solution. *[5 points]* There are many examples. One is $A = I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ so that $A + cB = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$, which has inverse $\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right)$.
