HOMEWORK 11

Due *4pm Wednesday, December 4.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. For each of the following 2×2 matrices A, find an invertible 2×2 matrix Q with real entries such that $Q^{-1}AQ$ has one of the three real canonical forms: diagonal $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, Jordan block $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$, or scaled rotation $q\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. Use this to compute A^6 by hand.

(a) $A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & 4 \\ -3 & 2 \end{pmatrix}$

2. Let λ be an eigenvalue of A. Say that v is a generalised eigenvector of order k if $(A - \lambda I)^k v = \mathbf{0}$ but $w_j := (A - \lambda I)^j v \neq \mathbf{0}$ for $1 \leq j < k$.

(a) Suppose v is a generalised eigenvector of order 2, so that $w_1 = (A - \lambda I)v$ is an eigenvector. Show that for every polynomial $q(t)$ we have $q(A)v = q(\lambda)v + q'(\lambda)w_1$.

(b) Now suppose v is a generalised eigenvector of order 3, so that $w_1 = (A - \lambda I)v$ is a generalised eigenvector of order 2, and $w_2 = (A - \lambda I)w_1$ is an eigenvector. Show that

$$A^N v = \lambda^N v + N\lambda^{N-1}w_1 + \frac{1}{2}N(N-1)\lambda^{N-2}w_2,$$

and use this to derive a formula for $q(A)v$.

3. Although every real matrix has eigenvalues in $\mathbb{C}$, we have seen examples of 2×2 matrices with no eigenvalues in $\mathbb{R}$. Show that such examples only exist in even dimensions. That is, show that if $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ and n is odd, then A has a real eigenvalue.

4. (a) Suppose $A, B \in \mathbb{M}_{n\times n}(\mathbb{C})$ and B is invertible. Show that there is $c \in \mathbb{C}$ such that $A + cB$ is not invertible. *Hint: look at* $\det(A + cB)$.

(b) Give an example of matrices A, B for which $A + cB$ is invertible for every $c \in \mathbb{C}$. *Hint: by the previous part, B must be non-invertible.*