Math 4377/6308 Advanced Linear Algebra I Fall 2013

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HOMEWORK 2 – solutions

Due *4pm Wednesday, September 4*. You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Determine whether or not each of the following is a subspace of $\mathbb{R}^2$. Justify your answer.

(a) $X_1 = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$

Solution. *[4 points]* Yes, X_1 is a subspace. Given any $(x, y), (x', y') \in X_1$ and $c \in \mathbb{R}$, we must check that $(cx + x', cy + y') \in X_1$. Indeed,

$$(cx + x') + (cy + y') = c(x + y) + (x' + y') = c \cdot 0 + 0 = 0.$$

(b) $X_2 = \{(x, y) \in \mathbb{R}^2 \mid x - 1 = 0\}$

Solution. *[4 points]* No, X_2 is not a subspace. It does not contain $(0, 0)$. (It also fails to be closed under addition or scalar multiplication.)

(c) $X_3 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$

Solution. *[4 points]* No, X_3 is not a subspace. It is not closed under addition: $(1, 0) \in X_3$ and $(0, 1) \in X_3$, but their sum $(1, 1)$ is not in X_3.

(d) $X_4 = \{(1, 0), (0, 1)\}$

Solution. *[4 points]* No, X_4 is not a subspace. It does not contain the zero vector. (It also fails to be closed under addition or scalar multiplication.)

(e) $X_5 = \text{span}\{(1, 0), (0, 1)\}$

Solution. *[4 points]* Yes, X_5 is a subspace; the span of any set of vectors is always a subspace.

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2. Prove that if X and Y are subspaces of V, then so are $X \cap Y$ and $X + Y$.

Solution. *[10 points]* Given any $v_1, v_2 \in X \cap Y$ and any $c \in K$, we have $v_1, v_2 \in X$ and $v_1, v_2 \in Y$ (by the definition of intersection). Thus the subspace property of X and Y implies that $cv_1 + v_2 \in X$ and $cv_1 + v_2 \in Y$, and in particular $cv_1 + v_2 \in X \cap Y$. Thus $X \cap Y$ satisfies the subspace property, and by Proposition 2.3 in the notes, it is a subspace.

For $X + Y$, we observe that given any $v_1, v_2 \in X + Y$ and $c \in K$, there exist $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $v_1 = x_1 + y_1$ and $v_2 = x_2 + y_2$. (This is from the definition of the sum of two subspaces.) Now $cv_1 + v_2 = c(x_1 + y_1) + (x_2 + y_2) = (cx_1 + x_2) + (cy_1 + y_2)$, and since X and Y are subspaces, we deduce that $cx_1 + x_2 \in X$ and $cy_1 + y_2 \in Y$, so their sum $cv_1 + v_2$ is in $X + Y$. Thus $X + Y$ is a subspace. (It is non-empty because both X and Y are.)

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3. Show that if $\mathbf{0} \in L \subset V$, then L is linearly dependent.

Solution. *[5 points]* The set $\{\mathbf{0}\}$ is linearly dependent because $1 \cdot \mathbf{0} = \mathbf{0}$. Thus L contains a linearly dependent set, hence by the previous exercise it is linearly dependent.

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4. Determine which of the following subsets of $\mathbb{R}^3$ are linearly independent.

(a) $S_1 = \{(1, 1, 0),\ (3, 0, 0)\}$

Solution. *[5 points]* If $a(1,1,0)+b(3,0,0)=\mathbf{0}$, then $a+3b=0$ (from the first coordinate) and $a=0$ (from the second), so we conclude that $a=b=0$, hence S_1 is linearly independent.

(b) $S_2 = \operatorname{span}\{(1,1,0),\ (3,0,0)\}$

Solution. *[5 points]* No, S_2 is not linearly independent. It contains the zero vector, so by the previous exercise it is linearly dependent.

(c) $S_3 = \{(2,0,1),\ (1,1,0),\ (0,0,1)\}$

Solution. *[5 points]* If $a(2,0,1)+b(1,1,0)+c(0,0,1)=\mathbf{0}$, then $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in the null space of $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Row reduction shows that the null space is trivial, so S_3 is linearly independent.

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5. Let V be a vector space, let $S \subset V$ be a spanning set, and let $L \subset V$ be linearly independent.

(a) Show that if $S \subset S' \subset V$, then S' is spanning.

Solution. *[5 points]* Given any $v \in V$, there are $x_1, \dots, x_n \in S$ and $c_1, \dots c_n \in K$ such that $\sum_{j=1}^n c_j x_j = v$. Because $x_j \in S'$ as well, this demonstrates that $v \in \operatorname{span}(S')$, and so S' is spanning.

(b) Show that if $L' \subset L$, then L' is linearly independent.

Solution. *[5 points]* If $x_1, \dots, x_n \in L'$ and $c_1, \dots, c_n \in K$ are such that $\sum_{j=1}^n c_j x_j = \mathbf{0}$, then we also have $x_1, \dots, x_n \in L$, and so linear independence of L implies that $c_1 = \cdots = c_n = \mathbf{0}$. This shows that L' is linearly independent.

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6. Prove that the set $\{\sin x, \cos x, \sin(2x)\} \subset C^1(\mathbb{R})$ is linearly independent. *Hint: if $a, b, c \in \mathbb{R}$ are such that $a \sin x + b \cos x + c \sin(2x) = 0$ for every $x \in \mathbb{R}$, then in particular, the equation is true for $x = 0, \pi/4, \pi/2$. Show that this implies that $a = b = c = 0$.*

Solution. *[10 points]* Suppose $a, b, c \in \mathbb{R}$ are such that $a \sin x + b \cos x + c \sin(2x) = 0$ for every $x \in \mathbb{R}$. Then in particular, for $x = 0$, $x = \pi/4$ and $x = \pi/2$ we get

$$a \cdot 0 + b \cdot 1 + c \cdot 0 = 0$$
$$a \cdot \frac{\sqrt{2}}{2} + b \cdot \frac{\sqrt{2}}{2} + c \cdot 1 = 0$$
$$a \cdot 1 + b \cdot 0 + c \cdot 0 = 0.$$

In other words, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in the null space of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Row reduction shows that the null space is trivial and so $a = b = c = 0$. Thus the set is linearly independent.