Math 4377/6308 Advanced Linear Algebra I Fall 2013
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HOMEWORK 3

Due *4pm Wednesday, September 11.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Consider the polynomials $f(x) = 2x^3 - x^2 + x + 3$, $g_1(x) = x^3 + x^2 + x + 1$, $g_2(x) = x^2 + x + 1$, and $g_3(x) = x + 1$. Determine (with proof) whether or not $f \in \text{span}\{g_1, g_2, g_3\}$.

Solution. *[5 points]* Expressing f as a linear combination of vectors of $\{g_1, g_2, g_3\}$ amounts to solving the equation

$$\begin{aligned} f(x) &= a_1 g_1(x) + a_2 g_2(x) + a_3 g_3(x) \\ &= a_1(x^3 + x^2 + x + 1) + a_2(x^2 + x + 1) + a_3(x + 1) \\ &= a_1 x^3 + (a_1 + a_2)x^2 + (a_1 + a_2 + a_3)x + (a_1 + a_2 + a_3). \end{aligned}$$

Comparing coefficients this becomes the system of equations

$$\begin{aligned} a_1 &= 2, \\ a_1 + a_2 &= -1, \\ a_1 + a_2 + a_3 &= 1, \\ a_1 + a_2 + a_3 &= 3. \end{aligned}$$

Subracting the third equation from the fourth (that is, comparing the linear and constant coefficients of the two polynomials in question) gives the inconsistent equation $0 = 2$, and so the system has no solutions. This means that f is **not** a linear combination of g_1, g_2, g_3.

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2. Consider the following three vectors in K^3: $u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Prove that $\{u_1, u_2, u_3\}$ is a basis for K^3.

Solution. *[10 points]* We must show that $\{u_1, u_2, u_3\}$ is linearly independent, and that it spans K^3.

Proof that it spans. To show that these three vectors generate K^3 it is enough to show that the equation $x = a_1u_1 + a_2u_2 + a_3u_3$ has a solution $a_1, a_2, a_3 \in K$ for every $x \in K^3$. Given $x = (x_1, x_2, x_3) \in K_3$, this equation becomes the system

$$a_1 + a_2 = x_1, \quad a_1 + a_3 = x_2, \quad a_2 + a_3 = x_3. \tag{1}$$

Subtracting the second equation from the first gives

$$a_2 - a_3 = x_1 - x_2,$$

and adding this to the third gives

$$2a_2 = x_1 - x_2 + x_3.$$

Similar manipulations let us solve for a_1, a_3, and we conclude that

$$a_1 = \frac{x_1 + x_2 - x_3}{2}, \qquad a_2 = \frac{x_1 - x_2 + x_3}{2}, \qquad a_3 = \frac{-x_1 + x_2 + x_3}{2} \tag{2}$$

solves the system and expresses x as a linear combination of u_1, u_2, u_3. In particular, these vectors span K^3.

Proof that it is linearly independent. The computation above solving the system (1) shows that if $a_1u_1 + a_2u_2 + a_3u_3 = \mathbf{0}$, then $a_1 = a_2 = a_3 = 0$. This follows from (2) and shows that $\{u_1, u_2, u_3\}$ is linearly independent.

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3. Let $L \subset V$ be linearly independent and let $v \in V \setminus L$. Show that $L \cup \{v\}$ is linearly independent if and only if $v \notin \operatorname{span} L$.

Solution. *[10 points]* We prove the equivalent statement that $L \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span} L$.

($\Leftarrow$). Suppose that $v \in \operatorname{span} L$. Then there are $v_1, \dots, v_n \in L$ and $c_1, \dots, c_n \in K$ such that $v = c_1v_1 + \cdots + c_nv_n$, and thus $c_1v_1 + \cdots + c_nv_n + (-1)v = \mathbf{0}$. Because $v \neq v_i$ for each $1 \leq i \leq n$, this gives a non-trivial representation of $\mathbf{0}$ as a linear combination of elements of $L \cup \{v\}$, so $L \cup \{v\}$ is linearly dependent.

($\Rightarrow$). Suppose $L \cup \{v\}$ is linearly dependent. Then there are $v_1, \dots, v_n \in L$ and $c_0, c_1, \dots, c_n \in K$ such that

$$c_0 v + c_1 v_1 + \cdots + c_n v_n = \mathbf{0}.$$

If $c_0 = 0$, then the above would be a non-trivial representation of $\mathbf{0}$ as a linear combination of vectors in L: because L is linearly independent, no such representation exists, and we conclude that $c_0 \neq 0$. In particular, we can solve for v to obtain

$$v = -c_1(c_0^{-1})v_1 - \cdots - c_n(c_0^{-1})v_n \in \operatorname{span} L.$$

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4. Suppose that $Y_1, \dots, Y_m$ are subspaces of V with the property that $V = Y_1 + \cdots + Y_m$. Show that the following are equivalent:

(a) every $v \in V$ can be written in a *unique* way as $v = y_1 + \cdots + y_m$, where $y_i \in Y_i$ for $1 \leq i \leq m$;

(b) if $y_i \in Y_i$ and $y_1 + \cdots + y_m = \mathbf{0}$, then $y_1 = \cdots = y_m = \mathbf{0}$.

Hint: This is very similar to Proposition 3.6 in the course notes.

Solution. *[10 points]* $(a) \Rightarrow (b)$. Note that $\mathbf{0} \in Y_i$ for every i, and in particular taking $y_i = \mathbf{0}$ gives a way to write $\mathbf{0}$ as a sum of elements of $Y_1, \dots, Y_m$. By property (a) this is the *only* way to write $\mathbf{0}$ as such a sum, and so $y_1 + \cdots + y_m = \mathbf{0}$ implies $y_1 = \cdots = y_m = \mathbf{0}$.

$(b) \Rightarrow (a)$. Suppose there is some $v \in V$ which can be written as $v = y_1 + \cdots + y_m$ and $v = z_1 + \cdots + z_m$, where $y_i, z_i \in Y_i$ for every i. Then we have

$$y_1 + \cdots + y_m = z_1 + \cdots + z_m,$$

and subtracting gives

$$(y_1 - z_1) + \cdots + (y_m - z_m) = \mathbf{0}.$$

Because Y_i is a subspace, we have $y_i - z_i \in Y_i$ for every i, and now property (b) implies that $y_i - z_i = \mathbf{0}$ for every i – that is, $y_i = z_i$. Thus the representation as $v = \sum_i y_i$ is unique for every $v \in V$.

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5. Let S_1 and S_2 be subsets of a vector space V. Show that $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

Solution. *[10 points]* To show that $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$ we show both inclusions.

($\subset$): Given $x \in \text{span}(S_1 \cup S_2)$, the definition of span implies that there exist $u_1, \cdots, u_n \in S_1 \cup S_2$ and $a_1, \cdots, a_n \in K$ such that $x = \sum_{i=1}^n a_i u_i$. Now each u_i is in either S_1 or S_2 (or both), so we can rewrite the set $\{u_1, \ldots, u_n\}$ as $\{v_1, \ldots, v_k, w_1, \ldots, w_m\}$, where $v_i \in S_1$ and $w_j \in S_2$. Similarly, rewrite the coefficients $\{a_1, \ldots, a_n\}$ as $\{b_1, \ldots, b_k, c_1, \ldots, c_m\}$, keeping each coefficient with the vector it was originally with, so that

$$x = a_1 u_1 + \cdots + a_n u_n = (b_1 v_1 + \cdots + b_k v_k) + (c_1 w_1 + \cdots + c_m w_m).$$

The sum in the first set of brackets is an element of $\text{span}(S_1)$, and the sum in the second set of brackets is an element of $\text{span}(S_2)$. This shows that x is an element of $\text{span}(S_1) + \text{span}(S_2)$, which establishes the first inclusion.

($\supset$): For the second inclusion, we fix an arbitrary $x \in \text{span}(S_1) + \text{span}(S_2)$ and note that x can be written as $x = y + z$ where $y \in \text{span}(S_1), z \in \text{span}(S_2)$. Now by definition of span we have

$$\begin{aligned} y &= b_1 v_1 + \cdots + b_k v_k, \\ z &= c_1 w_1 + \cdots + c_m w_m \end{aligned}$$

for some $v_1, \ldots, v_k \in S_1$, $w_1, \ldots, w_m \in S_2$, and $b_1, \ldots, b_k, c_1, \ldots, c_m \in K$. Adding the expressions for y and z together gives $x = y + z$ as a linear combination of elements of $S_1 \cup S_2$, hence $x \in \text{span}(S_1 \cup S_2)$, which completes the proof.

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6. Show that if $\{x, y\}$ is a basis for X, then so is $\{x + y, x - y\}$.

Solution. *[10 points]* First check that it spans. Given any $v \in X$, there exist $a, b \in K$ such that $v = ax + by$. (This is because $\{x, y\}$ is a basis.) We want to write $v = c(x + y) + d(x - y)$ for some $c, d \in K$.

This expands to $v = (c+d)x + (c-d)y$, and so we want c, d to solve

$$\begin{aligned} c + d &= a, \\ c - d &= b. \end{aligned}$$

Adding the equations gives $c = \frac{a+b}{2}$, subtracting them gives $d = \frac{a-b}{2}$. Since there is a solution for all $a, b \in K$, hence for all $v \in X$, we conclude that $\{x + y, x - y\}$ spans X.

Now we check that $\{x+y, x-y\}$ is linearly independent. If $c, d \in K$ are such that $c(x+y)+d(x-y) = \mathbf{0}$, then we have $(c+d)x+(c-d)y = \mathbf{0}$. By linear independence of $\{x, y\}$, this implies that $c + d = 0$ and $c - d = 0$. The second of these gives $d = c$, whence the first gives $2c = 0$, so that $d = c = 0$. This implies that $\{x + y, x - y\}$ is linearly independent. Thus it is a basis.

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7. Let $V = \{f \in \mathbb{P}_3 \mid f(1) = f(2) = 0\}$ be the set of cubic polynomials that vanish at the points 1 and 2.

(a) Show that V is a subspace of $\mathbb{P}_3$.

Solution. *[5 points]* If $f, g \in V$ and $c \in K$, then we check that $cf + g \in V$:

$$(cf + g)(1) = cf(1) + g(1) = c \cdot 0 + 0 = 0,$$

and similarly for $(cf + g)(2)$. Thus V is a subspace. (Note that it is non-empty because it contains the zero polynomial.)

(b) Determine $\dim V$ by finding a basis for V.

Solution. *[10 points]* Recall that a polynomial f has $f(1) = 0$ if and only if it has $(x-1)$ as a factor. Similarly with $(x-2)$, so we conclude that every $f \in V$ is of the form $f(x) = g(x)(x-1)(x-2)$ for some polynomial g.

Because $\deg f = 2 + \deg g \leq 3$, we see that g must be an element of $\mathbb{P}_1$. Thus $g(x) = ax + b$ for some $a, b \in K$. In particular, we claim that $f_1(x) = x(x-1)(x-2)$ (coming from $g(x) = x$) and

$f_2(x) = (x-1)(x-2)$ (coming from $g(x) = 1$) form a basis for V, so that $\dim V = 2$.

To see this, first observe that $(af_1 + bf_2)(x) = (ax+b)(x-1)(x-2)$, and by the above discussion every polynomial in V is of this form. Thus $V = \text{span}\{f_1, f_2\}$. Moreover, if $af_1 + bf_2$ is the zero polynomial, then we must have $a = b = 0$, so $\{f_1, f_2\}$ is linearly independent. Thus it is a basis for V.