Math 4377/6308 Advanced Linear Algebra I Fall 2013

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HOMEWORK 4 – solutions

Due *4pm Wednesday, September 18.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. $V = \mathbb{C}^n$ is naturally a vector space over $\mathbb{C}$, in which case it has dimension n, but it can also be viewed as a vector space over $\mathbb{R}$. Show that as a vector space over $\mathbb{R}$, the dimension of V is equal to $2n$.

Solution. *[10 points]* Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subset \mathbb{C}^n$ be the standard basis for $\mathbb{C}^n$. Then every element $v \in \mathbb{C}^n$ can be written in a unique way as

$$\begin{aligned} v &= c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n \\ &= (a_1 + ib_1)\mathbf{e}_1 + \cdots + (a_n + ib_n)\mathbf{e}_n, \end{aligned}$$

where $c_i \in \mathbb{C}$ and $a_i, b_i \in \mathbb{R}$. In particular, every $v \in \mathbb{C}^n$ can be written in a unique way as

$$v = a_1\mathbf{e}_1 + b_1(i\mathbf{e}_1) + a_2\mathbf{e}_2 + b_2(i\mathbf{e}_2) + \cdots + a_n\mathbf{e}_n + b_n(i\mathbf{e}_n),$$

which shows that $\{\mathbf{e}_1, i\mathbf{e}_1, \dots, \mathbf{e}_n, i\mathbf{e}_n\}$ is a basis for $\mathbb{C}^n$ over $\mathbb{R}$. (Note that this set is not linearly independent over $\mathbb{C}$.) This basis has $2n$ elements, which shows that the dimension of V as a vector space over $\mathbb{R}$ is $2n$.

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2. Let V be a vector space and let $W_1, W_2 \subset V$ be finite-dimensional subspaces. Recall from a previous assignment that $W_1 \cap W_2$ and $W_1 + W_2$ are also subspaces of V. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Hint: begin with a basis B for $W_1 \cap W_2$, and then extend B into a basis B_1 for W_1 and a basis B_2 for W_2.

Solution. *[20 points] An alternate proof using quotient spaces is given in Theorem 7 on page 10 of Lax's book.*

Let $B = \{w_1, \dots, w_k\}$ be a basis for $W_1 \cap W_2$, where $k = \dim(W_1 \cap W_2)$. By a result from the lecture, B can be extended to a basis

$B_1 = \{w_1, \ldots, w_k, v_1, \ldots, v_\ell\}$ for W_1, and $\dim(W_1) = k + \ell$. Similarly, B can be extended to a basis $B_2\{w_1, \ldots, w_k, u_1, \ldots, u_m\}$ for W_2, and $\dim(W_2) = k + m$.

We claim that $B_1 \cup B_2 = \{w_1, \ldots, w_k, v_1, \ldots, v_\ell, u_1, \ldots, u_m\}$ is a basis for $W_1 + W_2$. Once this is proven we will see that

$$\begin{aligned} &\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \\ &\qquad = (k + \ell) + (k + m) - k = k + \ell + m = \dim(W_1 + W_2). \end{aligned}$$

So we need only check that $B_1 \cup B_2$ is linearly independent and spans $W_1 + W_2$. To see that it is linearly independent, suppose $a_i, b_i, c_i \in K$ are such that

$$a_1 w_1 + \cdots + a_k w_k + b_1 v_1 + \cdots + b_\ell v_\ell + c_1 u_1 + \cdots + c_m u_m = \mathbf{0}.$$

Let $w = \sum_i a_i w_i$, and also consider $v = \sum_i b_i v_i$ and $u = \sum_i c_i u_i$. Note that $w \in W_1 \cap W_2$, $v \in W_1$, and $u \in W_2$. Moreover, we have $w + v + u = \mathbf{0}$. Thus $u = -(w + v) \in W_1$, and expanding this to $u = \sum_i (-a_i) w_i + \sum_i (-b_i) v_i$, we recall that since B_1 is a basis for W_1, every element of W_1 can be written in a unique way as a linear combination of elements of B_1. But by the above discussion, $u \in W_1 \cap W_2$ and so u can be written as $u = \sum_i a_i' w_i$, which implies that $b_i = 0$ for all i, and hence $v = 0$.

Now we have $\sum_i a_i w_i + \sum_i c_i u_i = \mathbf{0}$, and because B_2 is a basis, we conclude that $a_i = 0$ and $c_i = 0$ for all i. Thus $B_1 \cup B_2$ is linearly independent.

To show that $B_1 \cup B_2$ spans $W_1 + W_2$, choose any element $w \in W_1 + W_2$, and write $w = v + u$, where $v \in W_1$ and $u \in W_2$. Then since B_1 is a basis for W_1, there are coefficients a_i, b_i such that $v = \sum_i a_i w_i + \sum_i b_i v_i$. Similarly there are scalars c_i, d_i such that $u = \sum_i c_i w_i + \sum_i d_i u_i$. We get

$$\begin{aligned} w = v + u &= \left(\sum_i a_i w_i + \sum_i b_i v_i \right) + \left(\sum_i c_i w_i + \sum_i d_i u_i \right) \\ &= \sum_i (a_i + c_i) w_i + \sum_i b_i v_i + \sum_i d_i u_i \in \operatorname{span}(B_1 \cup B_2), \end{aligned}$$

which completes the proof.

3. Let W_1, W_2, X be subspaces of a vector space V. Is it necessarily true that $(W_1 + W_2) \cap X = (W_1 \cap X) + (W_2 \cap X)$? If it is true, prove it; if it is not true, find a counterexample.

Solution. *[10 points]* No, it is not necessarily true. For example, let $V = \mathbb{R}^2$ and consider the subspaces $W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$, $W_2 = \{(0, y) \mid y \in \mathbb{R}\}$, and $X = \{(a, a) \mid a \in \mathbb{R}\}$. Then $W_1 + W_2 = V$ and so $(W_1 + W_2) \cap X = X$, but $W_1 \cap X = W_2 \cap X = \{\mathbf{0}\}$ and so the right hand side is the trivial subspace.

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4. Let $n \geq 2$, and recall that $\mathbb{P}_n$ is the vector space of polynomials with real coefficients and degree at most n. Fix distinct real numbers $t_1, t_2, t_3 \in \mathbb{R}$ and let $X = \{f \in \mathbb{P}_n \mid f(t_1) = f(t_2) = f(t_3) = 0\}$. Show that X is a subspace of $\mathbb{P}_n$, and write down a basis for $\mathbb{P}_n/X$.

Solution. *[10 points]* First we show that X is a subspace. Clearly X contains the zero polynomial and so it is non-empty. Suppose $f, g \in X$ and $c \in \mathbb{R}$: then we have

$$(cf + g)(t_i) = cf(t_i) + g(t_i) = c \cdot 0 + 0 = 0$$

for $i = 1, 2, 3$, and so $cf + g \in X$. Thus X is a subspace.

Now we know from a result in lecture that $\dim(\mathbb{P}_n/X) = \dim(\mathbb{P}_n) - \dim(X) = n+1-\dim(X)$. So we need to determine $\dim(X)$. As on the previous assignment, we notice that $X = \{g(x)(x - t_1)(x - t_2)(x - t_3) \mid g \in \mathbb{P}_{n-3}\}$, and so $\dim(X) = n - 2$. We conclude that $\dim(\mathbb{P}_n/X) = 3$, and so $\{[1]_X, [x]_X, [x^2]_X\}$ will be a basis for $\mathbb{P}_n/X$ if we can show that it is linearly independent. To this end, suppose $a, b, c \in \mathbb{R}$ are such that $a[1]_X + b[x]_X + c[x^2]_X = [\mathbf{0}]_X$. This is equivalent to the statement that $a + bx + cx^2 \in X$, but we recall that any non-zero quadratic polynomial vanishes on at most two points, and so in order to have $f(x) = a + bx + cx^2 \in X$, which requires f to vanish at three points, we must have $a = b = c = 0$. Thus the set is linearly independent, hence it is a basis.

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5. With n, $\mathbb{P}_n$, and t_1, t_2, t_3 as in the previous problem, define $\ell_j(f) = f(t_j)$ for $f \in \mathbb{P}_n$ and $j = 1, 2, 3$.

(a) Show that ℓ_1, ℓ_2, ℓ_3 are all linear functions on $\mathbb{P}_n$.

Solution. *[5 points]* We check linearity by observing that for any $f, g \in \mathbb{P}_n$ and $c \in \mathbb{R}$, we have

$$\ell_j(cf + g) = (cf + g)(t_j) = cf(t_j) + g(t_j) = c\ell_j(f) + \ell_j(g).$$

(b) Show that $\{\ell_1, \ell_2, \ell_3\} \subset \mathbb{P}_n'$ is linearly independent.

Solution. *[10 points]* Suppose $a, b, c \in \mathbb{R}$ are such that $a\ell_1 + b\ell_2 + c\ell_3 = \mathbf{0}$, the zero functional. Then

$$a\ell_1(f) + b\ell_2(f) + c\ell_3(f) = af(t_1) + bf(t_2) + cf(t_3) = 0 \text{ for all } f \in \mathbb{P}_n.$$

Consider $f(t) = (t - t_1)(t - t_2)$, and note that $f(t_1) = f(t_2) = 0$, while $f(t_3) \neq 0$. Thus the above equation implies that $c = 0$. Similarly, applying the above equation to $f(t) = (t - t_1)(t - t_3)$ shows that $b = 0$, and $f(t) = (t - t_2)(t - t_3)$ shows that $a = 0$. We conclude that $a = b = c = 0$ whenever $a\ell_1 + b\ell_2 + c\ell_3 = \mathbf{0}$, and so $\{\ell_1, \ell_2, \ell_3\}$ is linearly independent.

(c) If $n = 2$, show that $\{\ell_1, \ell_2, \ell_3\}$ is a basis for $\mathbb{P}_2'$.

Solution. *[5 points]* We just saw that the set is linearly independent, and we know from Theorem 5.10 in the notes that $\dim(\mathbb{P}_2') = \dim(\mathbb{P}_2) = 3$. Thus any linearly independent set with 3 elements is a basis.