HOMEWORK 4

Due *4pm Wednesday, September 18.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. $V = \mathbb{C}^n$ is naturally a vector space over $\mathbb{C}$, in which case it has dimension n, but it can also be viewed as a vector space over $\mathbb{R}$. Show that as a vector space over $\mathbb{R}$, the dimension of V is equal to $2n$.

2. Let V be a vector space and let $W_1, W_2 \subset V$ be finite-dimensional subspaces. Recall from a previous assignment that $W_1 \cap W_2$ and $W_1 + W_2$ are also subspaces of V. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Hint: begin with a basis B for $W_1 \cap W_2$, and then extend B into a basis B_1 for W_1 and a basis B_2 for W_2.

3. Let W_1, W_2, X be subspaces of a vector space V. Is it necessarily true that $(W_1 + W_2) \cap X = (W_1 \cap X) + (W_2 \cap X)$? If it is true, prove it; if it is not true, find a counterexample.

4. Let $n \geq 2$, and recall that $\mathbb{P}_n$ is the vector space of polynomials with real coefficients and degree at most n. Fix distinct real numbers $t_1, t_2, t_3 \in \mathbb{R}$ and let $X = \{f \in \mathbb{P}_n \mid f(t_1) = f(t_2) = f(t_3) = 0\}$. Show that X is a subspace of $\mathbb{P}_n$, and write down a basis for $\mathbb{P}_n/X$.

Hint: An exercise on the previous assignment shows how to compute $\dim X$*, and then Theorem 5.7 from the notes tells how to find* $\dim(\mathbb{P}_n/X)$*, which tells you how many elements the basis should have.*

5. With $n, \mathbb{P}_n$, and t_1, t_2, t_3 as in the previous problem, define $\ell_j(f) = f(t_j)$ for $f \in \mathbb{P}_n$ and $j = 1, 2, 3$.

(a) Show that ℓ_1, ℓ_2, ℓ_3 are all linear functions on $\mathbb{P}_n$.

(b) Show that $\{\ell_1, \ell_2, \ell_3\} \subset \mathbb{P}_n'$ is linearly independent.
Hint: This requires showing that if $\sum_{i=1}^3 c_i \ell_i(f) = 0$ for every $f \in \mathbb{P}_n$, then $c_1 = c_2 = c_3 = 0$. Try using the specific examples $f(x) = (x - t_1)(x - t_2)$, $f(x) = (x - t_1)(x - t_3)$, and $f(x) = (x - t_2)(x - t_3)$.

(c) If $n = 2$, show that $\{\ell_1, \ell_2, \ell_3\}$ is a basis for $\mathbb{P}_2'$.