HOMEWORK 5 – solutions

Due *4pm Wednesday, September 25*. You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Determine (with proof) whether or not the following maps are linear.
(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (\sin(x), 2y)$.

Solution. *[5 points]* No, T is not linear. Consider $(x, y) = (\pi/2, 0)$ and $c = 2$, then

$$T(x, y) = T(\pi/2, 0) = (\sin(\pi/2), 0) = (1, 0),$$
$$T(c(x, y)) = T(\pi, 0) = (\sin(\pi), 0) = (0, 0) \neq 2T(x, y).$$

(b) $T\colon \mathbb{P}_4 \to \mathbb{R}^3$ given by $T(f) = (f(0), f(1), f(0))$.

Solution. *[5 points]* Yes, T is linear. Given $f, g \in \mathbb{P}_4$ and $c \in \mathbb{R}$, we have

$$\begin{aligned} T(cf + g) &= \big((cf + g)(0), (cf + g)(1), (cf + g)(0)\big) \\ &= \big(cf(0) + g(0), cf(1) + g(1), cf(0) + g(0)\big) \\ &= c(f(0), f(1), f(0)) + (g(0), g(1), g(0)) \\ &= cT(f) + T(g) \end{aligned}$$

(c) $T\colon C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $(Tf)(x) = f'(x) + \int_0^x f(y)\,dy$.

Solution. *[5 points]* Yes, T is linear. Given $f, g \in C^1(\mathbb{R})$ and $c \in \mathbb{R}$, we have

$$\begin{aligned} (T(cf + g))(x) &= \frac{d}{dx}(cf + g)(x) + \int_0^x (cf + g)(y)\,dy \\ &= c\frac{df}{dx}(x) + \frac{dg}{dx}(x) + c\int_0^x f(y)\,dy + \int_0^x g(y)\,dy \\ &= (cT(f) + T(g))(x). \end{aligned}$$

2. Let $T\colon V \to W$ be a linear map. Prove the following statements.
(a) If $X \subset V$ is a subspace of V, then $T(X)$ is a subspace of W.

Solution. *[5 points]* $T(X)$ is non-empty because X is non-empty. If w_1, w_2 are in $T(X)$ and $c \in K$, then there are $v_1, v_2 \in X$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Because X is a subspace of V, we have $cv_1 + v_2 \in X$. Thus we have $T(cv_1 + v_2) \in T(X)$. By linearity of T, we have

$$T(cv_1 + v_2) = cT(v_1) + T(v_2) = cw_1 + w_2,$$

and so $cw_1 + w_2 \in T(X)$, which confirms the condition for $T(X)$ to be a subspace of W.

(b) If $Y \subset W$ is a subspace of W, then $T^{-1}(Y)$ is a subspace of V.

Solution. *[5 points]* Because Y is a subspace, it contains $\mathbf{0}_W$, and because $T(\mathbf{0}_V) = \mathbf{0}_W$, we see that $\mathbf{0}_V \in T^{-1}(Y)$. Now suppose $v_1, v_2 \in T^{-1}(Y)$ and $c \in K$. Then by linearity of T we have

$$T(cv_1 + v_2) = cT(v_1) + T(v_2).$$

Because each of $T(v_1), T(v_2)$ is in Y, and Y is a subspace, the linear combination $cT(v_1) + T(v_2)$ is in Y as well, and thus $cv_1 + v_2 \in T^{-1}(Y)$. Thus $T^{-1}(Y)$ is a subspace of V.

3. Show that a linear map $T\colon V \to W$ is 1-1 if and only if $N_T = \{\mathbf{0}_V\}$.

Solution. *[10 points]* ($\Rightarrow$). This direction is straightforward: if T is 1-1, then every $w \in W$ has at most one pre-image $v \in V$, and so in particular $\mathbf{0}_W$ has at most one pre-image, which means that $N_T = T^{-1}(\mathbf{0}_W)$ has at most one element. Since N_T always contains $\mathbf{0}_V$, this implies that $N_T = \{\mathbf{0}_V\}$.

($\Leftarrow$). Suppose $N_T = \{\mathbf{0}_V\}$, and suppose $v_1, v_2 \in V$ are such that $T(v_1) = T(v_2)$. To show that T is 1-1 we must show that $v_1 = v_2$. Linearity of T implies that $T(v_1 - v_2) = T(v_1) - T(v_2) = \mathbf{0}_W$, hence $v_1 - v_2 \in N_T$. Because N_T is trivial, this implies that $v_1 - v_2 = \mathbf{0}_V$, which means that $v_1 = v_2$.

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4. Suppose that the vectors $v_1, \dots, v_n \in V$ are linearly independent, and that $T\colon V \to W$ is 1-1 and linear. Show that the vectors $T(v_1), \dots, T(v_n) \in W$ are linearly independent.

Solution. *[10 points]* Suppose $c_1, \dots, c_n \in K$ are such that $\sum_j c_j T(v_j) = \mathbf{0}_W$. We must show that $c_j = 0$ for all j. To this end, observe that linearity of T implies that

$$T\left(\sum_j c_j v_j\right) = \sum_j c_j T(v_j) = \mathbf{0}_W,$$

and because T is 1-1 this implies that $\sum_j c_j v_j = \mathbf{0}_V$. Now because $v_1, \dots, v_n$ are linearly independent, this implies that $c_j = 0$ for all j.

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5. Let $S \subset V$ be a spanning set, and suppose that $T\colon V \to W$ is onto and linear. Show that $T(S) \subset W$ is a spanning set for W.

Solution. *[10 points]* Let $w \in W$ be arbitrary. We must show that $w \in \operatorname{span}(T(S))$. Because T is onto, there exists $v \in V$ such that $T(v) = w$. Moreover, because S spans V there exist $c_1, \dots, c_n \in K$ and $v_1, \dots, v_n \in S$ such that $v = \sum_j c_j v_j$. Applying T and using linearity gives

$$w = T(v) = T\left(\sum_j c_j v_j\right) = \sum_j c_j T(v_j),$$

and so w can be written as a linear combination of $T(v_1), \dots, T(v_n) \in T(S)$, which implies that $w \in \operatorname{span}(T(S))$.

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6. Let $T\colon V \to W$ be a linear map, and recall that the transpose T' is a linear map $T'\colon W' \to V'$ that maps $\ell \in W'$ to $m \in V'$ given by $m(v) = \ell(Tv)$ for $v \in V$. (Here V' and W' are the dual spaces for V and W.) Show that if S and T are linear maps such that ST is defined, then $(ST)' = T'S'$.

Solution. *[10 points]* Suppose $T \in \mathbb{L}(V, W)$ and $S \in \mathbb{L}(W, X)$, so that ST is defined. Then by definition of the transpose, we have $T' \in \mathbb{L}(W', V')$ and $S' \in \mathbb{L}(X', W')$, so $T'S' \in \mathbb{L}(X', V')$ is defined. Note also that $ST \in \mathbb{L}(V, X)$, and so $(ST)' \in \mathbb{L}(X', V')$, so $(ST)'$ and $T'S'$ are linear maps with the same domain and the same target. Now we must show that they are equal.

To this end, choose any $\ell \in X'$. We want to show that $(ST)'(\ell) = (T'S')(\ell)$ as elements of V', which amounts to showing that

$$\big((ST)'(\ell)\big)(v) = \big((T'S')(\ell)\big)(v) \tag{1}$$

for every $v \in V$. By the definition of the transpose, the left-hand side of (1) is equal to

$$\big((ST)'(\ell)\big)(v) = \ell((ST)(v)) = \ell(S(T(v))),$$

while the right-hand side of (1) is equal to

$$\big((T'S')(\ell)\big)(v) = \big(T'(S'(\ell))\big)(v) = \big(S'(\ell)\big)(T(v)) = \ell(S(T(v))).$$

This confirms that the two sides are equal, so $(ST)' = T'S'$. One can also view the situation in terms of the following diagram.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[rr, bend left, "ST"] \arrow[dr, "(ST)'\ell=T'S'\ell\in V'"'] & W \arrow[r, "S"] \arrow[d, "S'\ell\in W'"] & X \arrow[dl, "\ell\in X'"] \\
& K &
\end{tikzcd}$$
