Dr. Vaughn Climenhaga, PGH 651A

HOMEWORK 5

Due *4pm Wednesday, September 25.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Determine (with proof) whether or not the following maps are linear.
(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (\sin(x), 2y)$.
(b) $T\colon \mathbb{P}_4 \to \mathbb{R}^3$ given by $T(f) = (f(0), f(1), f(0))$.
(c) $T\colon C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $(Tf)(x) = f'(x) + \int_0^x f(y)\,dy$.

2. Let $T\colon V \to W$ be a linear map. Prove the following statements.
(a) If $X \subset V$ is a subspace of V, then $T(X)$ is a subspace of W.
(b) If $Y \subset W$ is a subspace of W, then $T^{-1}(Y)$ is a subspace of V.

3. Show that a linear map $T\colon V \to W$ is 1-1 if and only if $N_T = \{\mathbf{0}_V\}$.

4. Suppose that the vectors $v_1, \dots, v_n \in V$ are linearly independent, and that $T\colon V \to W$ is 1-1 and linear. Show that the vectors $T(v_1), \dots, T(v_n) \in W$ are linearly independent.

5. Let $S \subset V$ be a spanning set, and suppose that $T\colon V \to W$ is onto and linear. Show that $T(S) \subset W$ is a spanning set for W.

6. Let $T\colon V \to W$ be a linear map, and recall that the transpose T' is a linear map $T'\colon W' \to V'$ that maps $\ell \in W'$ to $m \in V'$ given by $m(v) = \ell(Tv)$ for $v \in V$. (Here V' and W' are the dual spaces for V and W.) Show that if S and T are linear maps such that ST is defined, then $(ST)' = T'S'$.