Math 4377/6308 Advanced Linear Algebra I Fall 2013

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HOMEWORK 6 – solutions

Due *4pm Wednesday, October 9.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Construct two 2×2 matrices A and B such that $AB = 0$ but $BA \neq 0$.

Solution. *[10 points]* One example that works is $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. There are many others.

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2. Let $D \in \mathbb{M}_{n\times n}(K)$ be a *diagonal* matrix – that is, $D_{ij} = 0$ whenever $i \neq j$. Write $d_i = D_{ii}$ for the ith diagonal entry of D, and let $A \in \mathbb{M}_{n\times n}(K)$. Show that DA is obtained from A by muliplying the ith row of A by d_i, and AD is obtained from A by multiplying the jth column of A by d_j.

Solution. *[10 points]* A straightforward computation gives

$$(DA)_{ij} = \sum_{k=1}^{n} D_{ik} A_{kj} = D_{ii} A_{ij} = d_i A_{ij},$$

where the second equality follows since $D_{ik} = 0$ whenever $k \neq i$. The ith row of DA contains the entries $(DA)_{ij}$ for $1 \leq j \leq n$, and the above computation shows that these entries are obtained by multiplying the entries A_{ij} by d_i, which proves the claim about rows.

For columns we also see that

$$(AD)_{ij} = \sum_{k=1}^{n} A_{ik} D_{kj} = A_{ij} d_j,$$

since $D_{kj} = 0$ when $k = j$ and is equal to d_j when $k = j$. The jth column of AD contains the entries $(AD)_{ij}$ for $1 \leq i \leq n$, and we see that these are obtained by multiplying the entries A_{ij} by d_j.

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3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x,y) = (3x - y, 2x + 6y)$.

(a) Write the matrix of T relative to the standard basis on $\mathbb{R}^2$.

Solution. *[10 points]* We have $T\mathbf{e}_1 = T(1,0) = (3,2) = 3\mathbf{e}_1 + 2\mathbf{e}_2$, so the first column of the matrix is $\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)$. For the second column, we see that $T\mathbf{e}_2 = T(0,1) = (-1,6) = -\mathbf{e}_1 + 6\mathbf{e}_2$, so the second column is $\left(\begin{smallmatrix} -1 \\ 6 \end{smallmatrix}\right)$, and the matrix is $\left(\begin{smallmatrix} 3 & -1 \\ 2 & 6 \end{smallmatrix}\right)$,

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(b) Let $v_1 = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$, and write the matrix of T relative to the basis $\{v_1, v_2\}$.

Solution. *[10 points]* To find the first column we must write $T(v_1)$ as a linear combination of v_1 and v_2. We see that

$$T(v_1) = T\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 \\ -4 \end{smallmatrix}\right) = 4v_1 + 0v_2,$$

so the first column of the matrix is $\left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right)$. Similarly,

$$T(v_2) = T\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 5 \\ -10 \end{smallmatrix}\right) = 0v_1 + 5v_2,$$

so the second column is $\left(\begin{smallmatrix} 0 \\ 5 \end{smallmatrix}\right)$. Thus the matrix of T relative to $\{v_1, v_2\}$ is the diagonal matrix

$$\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}.$$

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4. Consider the bases $\{f_1, f_2, f_3, f_4\}$ and $\{g_1, g_2, g_3\}$ for $\mathbb{P}_3$ and $\mathbb{P}_2$, respectively, where

$$\begin{aligned} f_1(x) &= 2x, & g_1(x) &= 1 - x, \\ f_2(x) &= 1 - x^2, & g_2(x) &= 1 + x^2, \\ f_3(x) &= 2x + x^2, & g_3(x) &= x - x^2, \\ f_4(x) &= x^2 + 2x^3. \end{aligned}$$

Let $T\colon \mathbb{P}_3 \to \mathbb{P}_2$ be the linear map given by differentiation. For $1 \le j \le 4$, write Tf_j as a linear combination of g_1, g_2, g_3, and use this to write the matrix of T relative to the bases given above.

Solution. *[10 points]* The first step is to solve the linear equations generated by equating the coefficients of both sides of

$$\begin{aligned} f_i'(x) &= a_i(1-x) + b_i(1+x^2) + c_i(x-x^2) \\ &= (a_i+b_i) + (c_i-a_i)x + (b_i-c_i)x^2, \end{aligned}$$

which for $i = 1, 2, 3, 4$ yields

$$\begin{aligned} f_1'(x) &= 2 = (a_1+b_1) + (c_1-a_1)x + (b_1-c_1)x^2, \\ f_2'(x) &= -2x = (a_2+b_2) + (c_2-a_2)x + (b_2-c_2)x^2, \\ f_3'(x) &= 2+2x = (a_3+b_3) + (c_3-a_3)x + (b_3-c_3)x^2, \\ f_4'(x) &= 2x+6x^2 = (a_4+b_4) + (c_4-a_4)x + (b_4-c_4)x^2. \end{aligned}$$

The first has solution $a_1 = b_1 = c_1 = 1$. The second has solution $a_2 = 1, b_2 = c_2 = -1$. The third has solution $a_3 = 0, b_3 = c_3 = 2$. The fourth has solution $a_4 = -4, b_4 = 4, c_4 = -2$, and we conclude that

$$\begin{aligned} Tf_1 &= g_1 + g_2 + g_3, \\ Tf_2 &= g_1 - g_2 - g_3, \\ Tf_3 &= 2g_2 + 2g_3, \\ Tf_4 &= -4g_1 + 4g_2 - 2g_3. \end{aligned}$$

Thus relative to the bases given by f_j and g_i, the matrix of T is

$$\begin{pmatrix} 1 & 1 & 0 & -4 \\ 1 & -1 & 2 & 4 \\ 1 & -1 & 2 & -2 \end{pmatrix}$$

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5. Let V, W, X be vector spaces and let $S \in \mathbb{L}(V, W)$, $T \in \mathbb{L}(W, X)$, so the composition TS is in $\mathbb{L}(V, X)$.

(a) Prove that if TS is 1-1, then S is 1-1. Must T be 1-1?

Solution. *[7 points]* If S is not 1-1, then there are $v_1 \neq v_2 \in V$ such that $S(v_1) = S(v_2)$, and hence $TS(v_1) = TS(v_2)$, which shows that TS is not 1-1. On the other hand, it is not necessary for T to be 1-1: to see this, take $V = X = \mathbb{R}$ and $W = \mathbb{R}^2$, with $S(v) = (v, 0)$ and $T(x, y) = x$. We see that $TS(v) = T(v, 0) = v$, so TS is 1-1, but T is not 1-1.

(b) Prove that if TS is onto, then T is onto. Must S be onto?

Solution. *[7 points]* If T is not onto, then there is $x \in X$ such that $Tw \neq x$ for all $w \in W$. In particular, this implies that $TSv \neq x$ for all $v \in V$, since $Sv \in W$. Thus TS is not onto. On the other hand, it is not necessary for S to be onto. This follows from the same example as in the previous part, where $TS\colon \mathbb{R} \to \mathbb{R}$ is onto (indeed, it is the identity map), but $S\colon \mathbb{R} \to \mathbb{R}^2$ is not onto.

(c) Prove that if T and S are isomorphisms, then TS is an isomorphism.

Solution. *[6 points]* The composition of linear maps is linear, so TS is linear. We show that it is 1-1 and onto. Given any $v_1 \neq v_2 \in V$, we have $Sv_1 \neq Sv_2 \in W$, because S is 1-1, and so $TSv_1 \neq TSv_2 \in X$, because T is 1-1. This shows that TS is 1-1. To see that TS is onto, pick any $x \in X$ and use the fact that T is onto to find $w \in W$ such that $Tw = x$. Now because S is onto, there is $v \in V$ such that $Sv = w$, and therefore $TSv = Tw = x$. Thus TS is onto.