Math 4377/6308 Advanced Linear Algebra I Fall 2013

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HOMEWORK 7 – solutions

Due *4pm Wednesday, October 16.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Let $\beta = \{f_1, f_2, f_3\}$ and $\gamma = \{g_1, g_2, g_3\}$ be the bases for $\mathbb{P}_2$ given by

$$\begin{aligned} f_1(x) &= 2 + 2x - x^2 & g_1(x) &= 2 + x \\ f_2(x) &= 1 + x & g_2(x) &= -1 + x + 2x^2 \\ f_3(x) &= 1 + x^2 & g_3(x) &= 1 + x + x^2 \end{aligned}$$

Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{P}_2$, with $\mathbf{e}_1(x) = 1$, $\mathbf{e}_2(x) = x$, and $\mathbf{e}_3(x) = x^2$.

(a) Compute the change-of-coordinates matrix I_β^γ that turns β-coordinates into γ-coordinates, either directly or using the following steps.

(i) Find I_β^α and I_γ^α. (This requires almost no computation.)

(ii) Draw a commutative diagram showing that I_β^γ is the matrix satisfying $I_\gamma^\alpha I_\beta^\gamma = I_\beta^\alpha$.

(iii) Recall from your introductory linear algebra course that the matrix equation $AB = C$ can be solved for B by row reducing the augmented matrix $[A \mid C]$ to the form $[I \mid B]$.

(iv) Keeping this in mind, row reduce $[I_\gamma^\alpha \mid I_\beta^\alpha]$ to obtain $[I \mid I_\beta^\gamma]$.

Once you have computed I_β^γ, verify directly that $I_\gamma^\alpha I_\beta^\gamma = I_\beta^\alpha$.

Solution. *[15 points]* To do this directly, recall that $J = I_\beta^\gamma$ is the matrix whose coefficients are determined by

$$\begin{aligned} f_1 &= J_{11}g_1 + J_{21}g_2 + J_{31}g_3, \\ f_2 &= J_{12}g_1 + J_{22}g_2 + J_{32}g_3, \\ f_3 &= J_{13}g_1 + J_{23}g_2 + J_{33}g_3. \end{aligned}$$

These equations become

$$\begin{aligned} 2 + 2x - x^2 &= J_{11}(2 + x) + J_{21}(-1 + x + 2x^2) + J_{31}(1 + x + x^2), \\ 1 + x &= J_{12}(2 + x) + J_{22}(-1 + x + 2x^2) + J_{32}(1 + x + x^2), \\ 1 + x^2 &= J_{13}(2 + x) + J_{23}(-1 + x + 2x^2) + J_{33}(1 + x + x^2). \end{aligned}$$

Comparing coefficients in the first of these gives

$$\begin{aligned} 2 &= 2J_{11} - J_{21} + J_{31} && \text{(constant term)} \\ 2 &= J_{11} + J_{21} + J_{31} && \text{(linear term)} \\ -1 &= 2J_{21} + J_{31} && \text{(quadratic term)} \end{aligned}$$

which can be solved. One can find the remaining entries of J similarly, by comparing coefficients in the second and third equations above. Instead of giving details we describe the alternate solution sketched in the problem, which amounts to carrying out the same computations but in a more organised way.

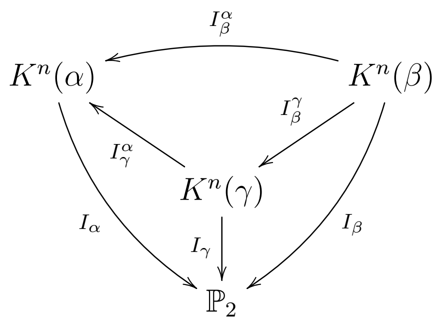
First, note that the columns of I_β^α are the coordinate representations of f_1, f_2, f_3 relative to the standard basis α, which are easy to determine because they are given by the coefficients of the polynomials. We have

$$[f_1]_\alpha = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix},\ [f_2]_\alpha = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ [f_3]_\alpha = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ \Rightarrow\ I_\beta^\alpha = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Similarly, we get

$$I_\gamma^\alpha = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

One possible commutative diagram relating the various change-of-coordinates matrices is



where we have written $K^n(\alpha)$ to denote the vector space K^n interpreted as coordinate representations relative to α, and similarly for $K^n(\beta)$ and $K^n(\gamma)$. This diagram illustrates that $I_\beta^\alpha = I_\gamma^\alpha I_\beta^\gamma$.

Thus finding $J = I_\beta^\gamma$ amounts to solving the matrix equation $AJ = B$, where $A = I_\gamma^\alpha$ and $B = I_\beta^\alpha$. Writing J in column form as

$J = [x_1 x_2 \cdots x_n]$, where $x_j \in K^n$, and writing B similarly as $B = [y_1 \cdots y_n]$, we see that this amounts to solving the equations $Ax_j = y_j$ for $1 \le j \le n$. Each of these can be solved by row reducing the augmented matrix $[A \mid y_j]$ to get $[I \mid x_j]$. (Note that we end up with the identity on the left because $A = I_\gamma^\alpha$ is invertible.)

We use the same sequence of row reductions for each j (because the left part of the matrix does not depend on j), and so we can write all n of these processes at once as

$$[A \mid B] = [A \mid y_1 \cdots y_n] \to [I \mid x_1 \cdots x_n] = [I \mid J],$$

where the arrow signifies row reduction. Thus in the present problem, to find J it suffices to row reduce

$$[I_\gamma^\alpha \mid I_\beta^\alpha] = \left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 2 & -3 \\ 0 & 1 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & -7 & -2 & 5 \end{array}\right)$$

where we have omitted the intermediate steps in the row reduction. We conclude that

$$J = I_\beta^\gamma = \begin{pmatrix} 6 & 2 & -3 \\ 3 & 1 & -2 \\ -7 & -2 & 5 \end{pmatrix},$$

and verify that

$$I_\gamma^\alpha I_\beta^\gamma = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 2 & -3 \\ 3 & 1 & -2 \\ -7 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = I_\beta^\alpha.$$

(b) Let $p(x) = x^2 + x$ and find $[p]_\alpha$. Compute $I_\alpha^\beta = (I_\beta^\alpha)^{-1}$, and use this together with I_β^γ to find $[p]_\beta$ and $[p]_\gamma$. Verify that $I_\gamma^\alpha [p]_\gamma = [p]_\alpha$.

Solution. *[10 points]* We can invert I_β^α by carrying out the row reduction $[I_\beta^\alpha \mid I] \to [I \mid (I_\beta^\alpha)^{-1}]$, which yields

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 & -1 & 0. \end{array}\right)$$

Now we see that $[p]_\alpha = \left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)$, and so

$$[p]_\beta = I_\alpha^\beta [p]_\alpha = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}.$$

This allows us to compute $[p]_\gamma$ by

$$[p]_\gamma = I_\beta^\gamma [p]_\beta = \begin{pmatrix} 6 & 2 & -3 \\ 3 & 1 & -2 \\ -7 & -2 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Finally, we check that

$$I_\gamma^\alpha [p]_\gamma = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = [p]_\alpha.$$

(c) Let $T \in \mathbb{L}(\mathbb{P}_2)$ be the differentiation operator, and find $[T]_\alpha$. Use I_α^β and I_β^α (which you computed in the previous parts) to find $[T]_\beta$.

Solution. *[10 points]* We have $T\mathbf{e}_1 = \mathbf{0}$, $T\mathbf{e}_2 = \mathbf{e}_1$, and $T\mathbf{e}_3 = 2\mathbf{e}_2$, so

$$[T]_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} [T]_\beta &= I_\alpha^\beta [T]_\alpha I_\beta^\alpha \\ &= \begin{pmatrix} 1 & -1 & -1 \\ -2 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & -2 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 & -2 \\ -10 & -2 & 6 \\ 4 & 1 & -2 \end{pmatrix} \end{aligned}$$

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2. Let A, B, C be invertible $n \times n$ matrices.

(a) Show that AB and BA are always conjugate.

Solution. *[5 points]* It suffices to observe that $A(BA)A^{-1} = AB$, so that AB can be obtained from BA via a conjugation by A.

(b) Use this to show that ABC and CAB are always conjugate.

Solution. *[5 points]* Similarly, $C(ABC)C^{-1} = CAB$.

(c) Give an example of invertible matrices A, B, C such that ABC and BAC are not conjugate. *Hint: Choose A, B such that $AB \neq BA$, and then let $C = (AB)^{-1}$.*

Solution. *[5 points]* Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $AB = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $BA = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $C = (AB)^{-1} = \left(\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $ABC = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $BAC = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, which is not conjugate to $ABC = I$ because for any invertible J we have $JIJ^{-1} = JJ^{-1} = I$, so the only matrix conjugate to the identity is the identity itself.

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3. Recall that a matrix $A \in \mathbb{M}_{n\times n}$ is *strictly upper triangular* if $A^{ij} = 0$ whenever $i \geq j$. Show that every strictly upper triangular matrix is nilpotent – that is, there exists k such that $A^k = 0$. *Hint: use induction to show that for every $k = 1, 2, 3, \dots,$ we have $(A^k)_{ij} = 0$ whenever $j \leq i + k - 1$.*

Solution. *[10 points]* As suggested, we use induction. For $k = 1$ the induction claim is exactly the hypothesis that A is strictly upper triangular. Now suppose A^k has the property that $(A^k)_{ij} = 0$ whenever $j \leq i + k - 1$. We prove the analogous property for A^{k+1}: given any i, j with $j \leq i + (k + 1) - 1 = i + k$, we have

$$(A^{k+1})_{ij} = \sum_{\ell=1}^{n} (A^k)_{i\ell} A_{\ell j}.$$

If $\ell \geq j$, then the fact that A is strictly upper triangular implies that $A_{\ell j} = 0$, and so the terms with $\ell \geq j$ in the above sum vanish, and we get

$$(A^{k+1})_{ij} = \sum_{\ell=1}^{j-1} (A^k)_{i\ell} A_{\ell j}.$$

But for $\ell \leq j-1$ we use $j \leq i+k$ to deduce that $\ell \leq i+k-1$, which by the inductive hypothesis implies that $(A^k)_{i\ell} = 0$, and hence these terms all vanish as well. Thus $(A^{k+1})_{ij} = 0$ whenever $j \leq i+k$, and by induction we have the result for all $k = 1, 2, 3, \dots$.

In particular, with $k = n$ we see that $j \leq i+n-1$ for *every* $1 \leq j \leq n$ and $1 \leq i \leq n$, and thus $(A^n)_{ij} = 0$ for all such i, j, meaning that A^n is the zero matrix.

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4. Suppose $T \in \mathbb{L}(V)$ satisfies the equation $T^2 = T$. Prove that $V = R_T \oplus N_T$. *Hint: this requires showing that $V = R_T + N_T$, and that $R_T \cap N_T = \{\mathbf{0}\}$. To show the first, it may help to prove that $v - T(v) \in N_T$ for all $v \in V$.*

Solution. *[10 points]* First we show that $V = R_T + N_T$. That is, for every $v \in V$ we produce $w \in R_T$ and $x \in N_T$ such that $v = w + x$. Let $w = T(v)$, then $w \in R_T$. Let $x = v - w$, then $v = w + x$, and it remains to show that $x \in N_T$. To this end, observe that

$$Tx = T(v-w) = Tv - Tw = Tv - T(Tv) = Tv - T^2 v = \mathbf{0}$$

where the last equality uses the fact that $T = T^2$. Thus $x \in N_T$ and we have shown that $V = R_T + N_T$.

It remains to show that $R_T \cap N_T = \{\mathbf{0}\}$. To this end, suppose $w \in R_T \cap N_T$. Because $w \in R_T$, we have $w = Tv$ for some $v \in V$. Then $Tw = T(Tv) = Tv = w$, where the second equality uses the fact that $T^2 = T$. But $w \in N_T$ implies that $Tw = \mathbf{0}$, and hence $w = \mathbf{0}$.
