HOMEWORK 7

Due *4pm Wednesday, October 16.* You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Let $\beta = \{f_1, f_2, f_3\}$ and $\gamma = \{g_1, g_2, g_3\}$ be the bases for $\mathbb{P}_2$ given by

$$\begin{aligned} f_1(x) &= 2 + 2x - x^2 & g_1(x) &= 2 + x \\ f_2(x) &= 1 + x & g_2(x) &= -1 + x + 2x^2 \\ f_3(x) &= 1 + x^2 & g_3(x) &= 1 + x + x^2 \end{aligned}$$

Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{P}_2$, with $\mathbf{e}_1(x) = 1$, $\mathbf{e}_2(x) = x$, and $\mathbf{e}_3(x) = x^2$.

(a) Compute the change-of-coordinates matrix I_β^γ that turns β-coordinates into γ-coordinates, either directly or using the following steps.
  (i) Find I_β^α and I_γ^α. (This requires almost no computation.)
  (ii) Draw a commutative diagram showing that I_β^γ is the matrix satisfying $I_\gamma^\alpha I_\beta^\gamma = I_\beta^\alpha$.
  (iii) Recall from your introductory linear algebra course that the matrix equation $AB = C$ can be solved for B by row reducing the augmented matrix $[A \mid C]$ to the form $[I \mid B]$.
  (iv) Keeping this in mind, row reduce $[I_\gamma^\alpha \mid I_\beta^\alpha]$ to obtain $[I \mid I_\beta^\gamma]$.

  Once you have computed I_β^γ, verify directly that $I_\gamma^\alpha I_\beta^\gamma = I_\beta^\alpha$.

(b) Let $p(x) = x^2 + x$ and find $[p]_\alpha$. Compute $I_\alpha^\beta = (I_\beta^\alpha)^{-1}$, and use this together with I_β^γ to find $[p]_\beta$ and $[p]_\gamma$. Verify that $I_\gamma^\alpha [p]_\gamma = [p]_\alpha$.

(c) Let $T \in \mathbb{L}(\mathbb{P}_2)$ be the differentiation operator, and find $[T]_\alpha$. Use I_α^β and I_β^α (which you computed in the previous parts) to find $[T]_\beta$.

2. Let A, B, C be invertible $n \times n$ matrices.

(a) Show that AB and BA are always conjugate.

(b) Use this to show that ABC and CAB are always conjugate.

(c) Give an example of invertible matrices A, B, C such that ABC and BAC are not conjugate. *Hint: Choose A, B such that $AB \neq BA$, and then let $C = (AB)^{-1}$.*

3. Recall that a matrix $A \in \mathbb{M}_{n\times n}$ is *strictly upper triangular* if $A^{ij} = 0$ whenever $i \geq j$. Show that every strictly upper triangular matrix is nilpotent – that is, there exists k such that $A^k = 0$. *Hint: use* *induction to show that for every* $k = 1, 2, 3, \ldots,$ *we have* $(A^k)_{ij} = 0$ *whenever* $j \leq i + k - 1$.

4. Suppose $T \in \mathbb{L}(V)$ satisfies the equation $T^2 = T$. Prove that $V = R_T \oplus N_T$. *Hint: this requires showing that* $V = R_T + N_T$, *and that* $R_T \cap N_T = \{\mathbf{0}\}$. *To show the first, it may help to prove that* $v - T(v) \in N_T$ *for all* $v \in V$.