HOMEWORK 8

Due *4pm Wednesday, October 23*. You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. Let $T \in \mathbb{L}(V)$ be nilpotent and let $P \in \mathbb{L}(V)$ be a projection.

(a) Show that 0 is an eigenvalue of T, and that it is the only eigenvalue.

Solution. *[10 points]* First observe that if $v \in V$ is any non-zero vector, then by nilpotence there is k such that $T^k v = \mathbf{0}$. Assume that k is the smallest such value – that is, $T^{k-1}v \neq \mathbf{0}$ and $T^k v = \mathbf{0}$. Then $w = T^{k-1}v$ has the property that $w \neq \mathbf{0}$ and $Tw = \mathbf{0} = 0w$, thus w is an eigenvector for the eigenvalue 0, so 0 is an eigenvalue of T.

Now suppose $\lambda \neq 0$ is an eigenvalue of T. Then there is $v \in V$ such that $v \neq \mathbf{0}$ and $Tv = \lambda v$. In particular, for every $k \in \mathbb{N}$ we have $T^k v = \lambda^k v \neq \mathbf{0}$, which contradicts the definition of a nilpotent operator. It follows that T has no non-zero eigenvalues.

(b) Show that the only possible eigenvalues of P are 0 and 1. Show that if P is not the identity or the zero transformation, then 0 and 1 are both eigenvalues.

Solution. *[10 points]* Suppose λ is an eigenvalue of P, and let v be an eigenvector for λ. Then because $P^2 = P$, we see that $P^2 v = Pv$, and because v is an eigenvector, we have $P^2 v = \lambda^2 v$ and $Pv = \lambda v$. It follows that $\lambda^2 v = \lambda v$, so $(\lambda^2 - \lambda)v = \mathbf{0}$. Because $v \neq \mathbf{0}$, this implies that $\lambda^2 - \lambda = \lambda(\lambda - 1)0$, so $\lambda = 0$ or $\lambda = 1$. Thus every eigenvalue is equal to 0 or 1.

Note that every non-trivial vector in N_P is an eigenvector for 0, and so 0 is an eigenvalue if and only if N_P is non-trivial. Similarly, if $w \in R_P$, then there is $v \in V$ such that $Pv = w$, and hence $Pw = P(Pv) = P^2 v = Pv = w$ using the fact that $P^2 = P$. Thus every element of R_P is an eigenvector for 1, and so 1 is an eigenvalue if and only if R_P is non-trivial.

Recall that for a projection, $V = R_P \oplus N_P$. If $P \neq \mathbf{0}$ then $N_P \neq V$, so R_P is non-trivial and 1 is an eigenvalue. The previous paragraph shows that if $R_P = V$, then $P = I$, so if P is not the identity then $R_P \neq V$ and N_P is non-trivial, so 0 is an eigenvalue.

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2. Consider the matrix $A = \begin{pmatrix} -4 & -6 & 3 \\ 2 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$

(a) Show that $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = 2$ are eigenvalues of A by finding eigenvectors $v_1, v_2, v_3 \in \mathbb{R}^3$ such that $Av_j = \lambda_j v_j$.

Solution. *[10 points]* Note that v_j is an eigenvector for λ_j if and only if it is a non-trivial solution to the homogeneous linear equation $(A - \lambda_j I)v_j = \mathbf{0}$. Thus we can find v_1 by row reducing A, we can find v_2 by row reducing $A - (-1)I = A + I$, and we can find v_3 by row reducing $A - 2I$. These row reductions proceed as shown (omitting several intermediate steps):

$$A = \begin{pmatrix} -4 & -6 & 3 \\ 2 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & -1 \\ 2 & 4 & -2 \\ 0 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A + I = \begin{pmatrix} -3 & -6 & 3 \\ 2 & 5 & -2 \\ -2 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A - 2I = \begin{pmatrix} -6 & -6 & 3 \\ 2 & 2 & -2 \\ -2 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We conclude that $Av_1 = \mathbf{0}$ has the non-trivial solution $v_1 = \left(\begin{smallmatrix} 0 \\ 1 \\ 2 \end{smallmatrix}\right)$, that $Av_2 = -v_2$ has the non-trivial solution $v_2 = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$, and $Av_3 = 2v_3$ has the non-trivial solution $v_3 = \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right)$.

Note that we could replace v_j with av_j for any scalar $a \neq 0$ and still obtain an eigenvector.

(b) Let $Q = [v_1 \mid v_2 \mid v_3] \in \mathbb{M}_{3\times 3}$ be the matrix whose column vectors are the eigenvectors v_j. Compute $Q^{-1}AQ$, and verify that this is a diagonal matrix whose diagonal entries are the eigenvalues of A.

Solution. *[10 points]* We start by computing Q^{-1}, by row reducing $[Q|I]$ to $[I|Q^{-1}]$:

$$[Q|I] = \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right)$$

Now we have

$$Q^{-1}AQ = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -4 & -6 & 3 \\ 2 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Notice that the columns of AQ are scalar multiples of the columns of Q, and that the ratios between them are the eigenvalues. This is because the columns of Q were chosen to be eigenvectors of A.

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3. Let x_n be the sequence of integers defined by $x_0 = x_1 = x_2 = 1$ and the recursive relationship

$$x_n = x_{n-1} + 2x_{n-2} - x_{n-3}.$$

Write down a 3×3 matrix A whose powers can be used to compute x_n, and explain how x_n can be obtained from A^n. *Hint: Let $v_n \in \mathbb{R}^3$ be the* *vector with components x_n, x_{n-1}, x_{n-2}, and find a relationship between v_n and v_{n-1}.*

Solution. *[10 points]* As suggested, let $v_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{pmatrix}$ for $n \geq 2$. Then $v_{n-1} = \begin{pmatrix} x_{n-1} \\ x_{n-2} \\ x_{n-3} \end{pmatrix}$, and to express v_n in terms of v_{n-1} we need to use the relationship $x_n = x_{n-1} + 2x_{n-2} - x_{n-3}$ to get

$$v_n = \begin{pmatrix} x_{n-1} + 2x_{n-2} - x_{n-3} \\ x_{n-1} \\ x_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \\ x_{n-3} \end{pmatrix}$$

Note that x_n is the first component of v_n, and that with $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, the relationship $v_n = Av_{n-1}$ holds for all $n \geq 3$. Thus we have

$$v_n = A^{n-2} v_2 = A^{n-2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

In particular, x_n is the sum of the entries in the first row of A^{n-2}.

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4. Use the formula for determinant of a 2×2 matrix to prove that $\det(AB) = \det(A)\det(B)$ for any $A, B \in \mathbb{M}_{2\times 2}$.

Solution. *[10 points]* Consider the 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. On the one hand we have

$$\begin{aligned} \det(A)\det(B) &= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ &= (ad - bc)(wz - xy) \\ &= adwz - bcwz - adxy + bcxy. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \det(AB) = \det &\begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} \\ &= (aw + by)(cx + dz) - (ax + bz)(cw + dy) \\ &= (awcx + bycx + awdz + bydz) \\ &\qquad - (axcw + bzcw + axdy + bzdy) \\ &= bycx + axdz - bzcw - axdy, \end{aligned}$$

and we see that the two expressions are equal.

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5. Given each of the following sets of four points in $\mathbb{R}^2$, compute the area of the parallelogram with vertices at those four points.

(a) $(0,0)$, $(2,1)$, $(1,1)$, and $(3,2)$
(b) $(0,0)$, $(3,2)$, $(2,4)$, and $(-1,2)$
(c) $(1,1)$, $(2,2)$, $(0,3)$, and $(-1,2)$

Solution. *[10 points]*

(a) Let $v = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, then the vertices are 0, v, w, and $v+w$, so the area of the parallelogram is $\det\left(\begin{smallmatrix} v & w \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) = 2 \cdot 1 - 1 \cdot 1 = 1$.

(b) Let $v = \left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right)$, then the vertices are 0, v, w, and $v+w$, so the area of the parallelogram is $\det\left(\begin{smallmatrix} v & w \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} 3 & -1 \\ 2 & 2 \end{smallmatrix}\right) = 3 \cdot 2 - (-1) \cdot 2 = 8$.

(c) Let $a = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and $w = \left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)$: then then vertices of the parallelogram are a, $a+v$, $a+w$, and $a+v+w$, so it has the same area as the paralellogram with vertices 0, v, w, and $v+w$, which is given by $\det\left(\begin{smallmatrix} v & w \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} 1 & -2 \\ 1 & 1 \end{smallmatrix}\right) = 1 \cdot 1 - (-2) \cdot 1 = 3$.
