HOMEWORK 9 – solutions

Due *4pm Wednesday, November 13*. You will be graded not only on the correctness of your answers but also on the clarity and completeness of your communication. Write in complete sentences.

1. (a) Let $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ be diagonal. Show that $\det D = \prod_{i=1}^n \lambda_i$.

Solution. *[5 points]* Using a sum over permutations,

$$\det D = \sum_{\pi \in S_n} \operatorname{sgn} \pi \prod_{i=1}^n D_{i,\pi(i)}.$$

Because $D_{ij} = 0$ whenever $i \neq j$, the only non-zero term occurs when $\pi(i) = i$ for all i, which only happens with the identity permutation. The identity permutation π has $\operatorname{sgn} \pi = 1$, and so $\det D = \prod_{i=1}^n D_{ii} = \prod \lambda_i$. *This proof can also be done using cofactor expansion, or another characterisation of determinant.*

(b) Let $A \in \mathbb{M}_{n\times n}$ be upper triangular, and show that $\det A = \prod_{i=1}^n A_{ii}$.

Solution. *[5 points]* We use cofactor expansion and go by induction. *The proof can also be given via a sum over permutations, as above, or via other methods. We use cofactor expansion here to illustrate a different approach, not because one approach is better.*

Suppose the statement is true for $n - 1$. By cofactor expansion along the first column,

$$\det A = \sum_{i=1}^n (-1)^{i+1} A_{i1} \det \tilde{A}_{i1}.$$

Because A is upper triangular, $A_{i1} = 0$ for all $i > 1$, so we get

$$\det A = A_{11} \det \tilde{A}_{11}.$$

Now we observe that $\tilde{A}_{11}$ is an upper triangular $(n - 1) \times (n - 1)$ matrix, whose diagonal entries are precisely $A_{22}, A_{33}, \dots, A_{nn}$.

By the inductive hypothesis we get

$$\det \tilde{A}_{11} = \prod_{j=2}^{n} A_{jj},$$

and the result for A follows. By induction, the statement is true for all n.

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2. Use row reduction to compute the determinant of $A = \begin{pmatrix} 0 & 4 & -1 & 1 \\ -3 & 1 & 1 & 2 \\ 1 & 0 & -2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix}$.

Solution. *[15 points]* Add 3 times the third row to the second row, and subtract 2 times the third row from the fourth row, to get

$$\begin{pmatrix} 0 & 4 & -1 & 1 \\ -3 & 1 & 1 & 2 \\ 1 & 0 & -2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 4 & -1 & 1 \\ 0 & 1 & -5 & 11 \\ 1 & 0 & -2 & 3 \\ 0 & 3 & 4 & -5 \end{pmatrix}$$

Now subtract 4 times the second row from the first row, and 3 times the second row from the fourth row, to get

$$\begin{pmatrix} 0 & 4 & -1 & 1 \\ 0 & 1 & -5 & 11 \\ 1 & 0 & -2 & 3 \\ 0 & 3 & 4 & -5 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 19 & -43 \\ 0 & 1 & -5 & 11 \\ 1 & 0 & -2 & 3 \\ 0 & 0 & 19 & -38 \end{pmatrix}$$

Subtract the first row from the fourth row to get

$$\begin{pmatrix} 0 & 0 & 19 & -43 \\ 0 & 1 & -5 & 11 \\ 1 & 0 & -2 & 3 \\ 0 & 0 & 19 & -38 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 19 & -43 \\ 0 & 1 & -5 & 11 \\ 1 & 0 & -2 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Swap the first and third rows to get

$$\begin{pmatrix} 0 & 0 & 19 & -43 \\ 0 & 1 & -5 & 11 \\ 1 & 0 & -2 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

which is upper triangular. Call this last matrix B. Going from A to B involved row operations of Types 1 and 2 only. There was one operation of Type 2 (transposition), so $\det A = -\det B$. Because B is upper triangular, $\det B = 1 \cdot 1 \cdot 19 \cdot 5 = 95$, and we conclude that $\det A = -95$.

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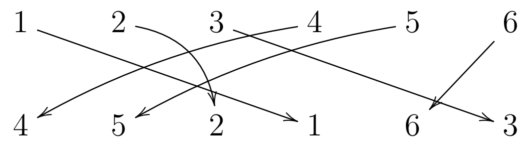
3. Determine the signs of the following permutations on 6 symbols.

(a) $\pi = (2, 1, 3, 4, 5, 6)$

Solution. *[5 points]* The ordering $(2, 1, 3, 4, 5, 6)$ is obtained from $(1, 2, 3, 4, 5, 6)$ by a single transposition (swap the 1 and 2), so $\operatorname{sgn} \pi = -1$.

(b) $\pi = (4, 5, 2, 1, 6, 3)$

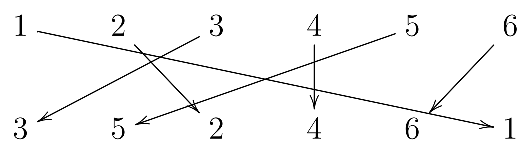
Solution. *[5 points]* The crossing diagram



has 8 intersections, so $\operatorname{sgn} \pi = (-1)^8 = 1$.

(c) $\pi = (3, 5, 2, 4, 6, 1)$

Solution. *[5 points]* The crossing diagram



has 8 intersections, so $\operatorname{sgn} \pi = (-1)^8 = 1$.

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4. Let $A \in \mathbb{M}_{n \times n}$.

(a) Show that $\det(\lambda A) = \lambda^n \det A$ for any $\lambda \in K$.

Solution. *[10 points]* Using sums over permutations,

$$\begin{aligned}\det(\lambda A) &= \sum_{\pi \in S_n} (\operatorname{sgn} \pi) \prod_{i=1}^{n} (\lambda A)_{i,\pi(i)} \\ &= \sum_{\pi \in S_n} (\operatorname{sgn} \pi) \prod_{i=1}^{n} \lambda(A_{i,\pi(i)}) \\ &= \lambda^n \sum_{\pi \in S_n} (\operatorname{sgn} \pi) \prod_{i=1}^{n} A_{i,\pi(i)} = \lambda^n \det A.\end{aligned}$$

This can also be done using cofactor expansion and induction, or via row operations using multilinearity of determinant.

(b) Say that A is *skew-symmetric* if $A^t = -A$. Suppose that A is skew-symmetric and invertible. Show that n is even.

Solution. *[10 points]* By the previous part, any $n \times n$ matrix A has

$$\det(-A) = (-1)^n \det A.$$

On the other hand, skew-symmetry implies

$$\det(-A) = \det A^t = \det A.$$

Thus $\det A = (-1)^n \det A$, and invertibility of A implies that $\det A \neq 0$, so $(-1)^n = 1$. This implies that n is even.

5. Prove that $\lambda \in K$ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Solution. *[10 points]* ($\Rightarrow$). If λ is an eigenvalue, then there is $v \in K^n$ such that $v \neq \mathbf{0}$ and $Av = \lambda v$. This gives $Av - \lambda v = \mathbf{0}$, so $(A - \lambda I)v = \mathbf{0}$. This means that $A - \lambda I$ has non-trivial nullspace, hence is non-invertible, hence has zero determinant.

($\Leftarrow$). If $\det(A - \lambda I) = 0$, then $A - \lambda I$ is non-invertible, so it has non-trivial nullspace. Any non-trivial element of this nullspace is an eigenvector for λ, hence λ is an eigenvalue.