Test 1 Review

You are responsible for knowing the definitions of all the terms below. Also listed are some common procedures you should know how to do (e.g. "determining whether or not a subset is a subspace") and some general results you should know (e.g. "If S is spanning and L is linearly independent, then $\#L \leq \#S$"). You may be asked to prove simple instances of these results (e.g. "If there exists $v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$, then S is linearly dependent").

Vector spaces.

- Examples: $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{P}_n$, $\mathbb{P}$, $C(\mathbb{R})$, $C^1(\mathbb{R})$.
- Determining whether something is or is not a vector space

Subspaces.

- Examples, determining whether something is a subspace.
- Sums of subspaces.
- Direct sums, determining whether or not a given sum is direct.
- Existence of a complement of a subspace W in finite-dimensional spaces (another subspace X such that $V = W \oplus X$) – Theorem 4.11

Linear combinations and span.

- Determining whether or not a vector is in the span of a set.
- Determining whether or not a set spans the entire vector space.
- Relationship to non-homogeneous systems of linear equations.

Linear dependence and independence.

- Determining whether or not a given set is linearly dependent.
- If S is spanning and L is linearly independent, then $\#L \leq \#S$ – Proposition 3.4
- Behaviour of linearly dependent and independent sets under adding or removing vectors – Exercise 3.3.
- S is linearly dependent iff there is $v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$, iff there is $v \in S$ such that $\operatorname{span}(S \setminus \{v\}) = \operatorname{span}(S)$ – Proposition 3.5
- S is linearly independent iff everything in $\operatorname{span} S$ can be represented in a *unique* way – Proposition 3.6

Bases and dimension

- Every basis has the same number of vectors – Theorem 3.9
- Finite spanning set implies existence of a finite basis – Lemma 4.5
- Every linearly independent set can be completed to a basis if V is finite-dimensional – Theorem 4.7

- Determining whether a given set is a basis. If the number of vectors in B is not equal to $\dim V$, then B is not a basis. If it is equal, then it is only necessary to check either linear independence or spanning, and the other follows.

Quotient spaces

- Equivalence modulo a subspace, congruence classes.
- Congruence classes are affine subspaces: $[v]_Y = v + Y$. (Proposition 5.4)
- Formula for dimension of quotient space – Theorem 5.7

Dual spaces

- Linear functionals, examples.
- Finite-dimensionality implies $\dim(V') = \dim(V)$ – Theorem 5.10

Linear maps

- Examples: differentiation, evaluation, matrix multiplication, rotation.
- Determining whether a map is linear.
- Behaviour of linear dependence and independence: a dependent set gets mapped to a dependent set (Proposition 6.13). An independent set gets mapped to an independent set if the map is 1-1 (Exercise 6.14).
- A spanning set gets mapped to a spanning set if T is onto – Exercise 6.15
- Images and preimages of subspaces are subspaces – Theorem 6.16
- A linear transformation is uniquely determined by its action on a basis – Lemma 7.7

Nullspace and range

- Determining the nullspace and range of a map.
- T is 1-1 iff N_T is trivial – Exercise 7.4
- Solution sets of systems of equations and ODEs as nullspaces.
- Dimension of nullspace + dimension of range = dimension of domain (Theorem 9.1)

Isomorphisms

- If T is an isomorphism, then B is a basis if and only if $T(B)$ is – Proposition 7.6
- Two finite-dimensional vector spaces are isomorphic iff they have the same dimension – Theorem 7.8

Compositions of linear maps

- Associativity and distributivity
- Non-commutativity.
- Transpose of a linear transformation, relationship to transpose of a matrix.