Math 4377/6308 Advanced Linear Algebra I Fall 2013

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MIDTERM TEST #1

Monday, September 30, 2013

You must give complete justification for all answers in order to receive full credit.

Name: SOLUTIONS

	Points	Possible
Problem 1		/25
Problem 2		/25
Problem 3		/30
Problem 4		/20
Total		/100

1. Let V be a vector space over a field K.

(a) Let $W_1, W_2 \subset V$ be subspaces, and define what it means to have $V = W_1 \oplus W_2$. [*5 points*]

$V = W_1 \oplus W_2$ if $V = W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$

and $W_1 \cap W_2 = \{\vec{0}\}$.

(Alternate & perhaps better definition: $V = W_1 + W_2$ and

if $w_1 \in W_1$, $w_2 \in W_2$ are such that $w_1 + w_2 = \vec{0}$,

then $w_1 = w_2 = \vec{0}$)

(b) Define what it means for $x_1, \ldots, x_n \in V$ to be linearly independent. [*5 points*]

$x_1, \ldots, x_n$ are linearly independent if

$c_1 x_1 + \cdots + c_n x_n = \vec{0}$, $c_1, \ldots, c_n \in K$ $\Longrightarrow$ $c_1 = \cdots = c_n = 0$

(c) State what it means for V to be finite-dimensional. Assuming V is finite-dimensional, define the dimension of V. [*5 points*]

V is finite-dimensional if it has a finite basis. The dimension of V is the cardinality of (number of elements in) any basis for V.

(d) Define the dual space V'. [*5 points*]

The dual space V' is the vector space whose elements are all linear functionals $\ell : V \to K$.

(e) Let W be another vector space over K. Define what it means for $T: V \to W$ to be an isomorphism. [*5 points*]

$T: V \to W$ is an isomorphism if it is linear, 1-1, and onto.

2. (a) Let $W = \{(x, y) \in \mathbb{R}^2 \mid x - xy = 0\}$. Is W a subspace of $\mathbb{R}^2$? Prove your answer. *[10 points]*

No, W is not a subspace. Notice that

$$W = \{(x,y) \mid x(1-y) = 0\} = \{(x,y) \mid x = 0\} \cup \{(x,y) \mid y = 1\}.$$

In particular, $(1,1) \in W$ but $2 \cdot (1,1) = (2,2) \notin W$.

(b) Let V and W be vector spaces and let $T \colon V \to W$ be a linear transformation. Let Y be a subspace of W. Show that $T^{-1}(Y)$ is a subspace of V. *[15 points]*

$T^{-1}(Y)$ is non-empty because it contains $\vec{0}_V$ (this is because $\vec{0}_W \in Y$ and $T(\vec{0}_V) = \vec{0}_W$).

Suppose $x, y \in T^{-1}(Y)$ and $c \in K$. Then $T(x), T(y) \in Y$, and because Y is a subspace we have $cT(x) + T(y) \in Y$.

Now by linearity of T,

$$T(cx + y) = cT(x) + T(y) \in Y,$$

hence $cx + y \in T^{-1}(Y)$. Thus $T^{-1}(Y)$ is a subspace.

3. (a) Let V be a vector space and let $v_1, \ldots, v_n \in V$ be linearly independent. Given $v_{n+1} \in V$, show that $v_1, \ldots, v_{n+1}$ are linearly independent if and only if $v_{n+1} \notin \operatorname{span}\{v_1, \ldots, v_n\}$. *[15 points]*

($\Rightarrow$) If $v_{n+1} \in \operatorname{span}\{v_1, \ldots, v_n\}$, then $\exists\, c_1, \ldots, c_n \in K$ s.t. $v_{n+1} = c_1 v_1 + \cdots + c_n v_n$. This yields

$$c_1 v_1 + \cdots + c_n v_n - v_{n+1} = \vec{0},$$

and $-1 \neq 0$ so this is a non-trivial linear combination, hence $v_1, \ldots, v_{n+1}$ are linearly dependent.

($\Leftarrow$) If $v_1, \ldots, v_{n+1}$ are linearly dependent, then $\exists\, c_1, \ldots, c_{n+1} \in K$ s.t. not all the c_i are 0, and

$$(*) \quad c_1 v_1 + \cdots + c_n v_n + c_{n+1} v_{n+1} = \vec{0}.$$

If $c_{n+1} = 0$ then some other c_i must be $\neq 0$, and so $v_1, \ldots, v_n$ are dependent, contradicting the hypothesis. Thus $c_{n+1} \neq 0$, and $(*)$ implies

$$v_{n+1} = -\frac{c_1}{c_{n+1}} v_1 - \frac{c_2}{c_{n+1}} v_2 - \cdots - \frac{c_n}{c_{n+1}} v_n$$

$\therefore\ v_{n+1} \in \operatorname{span}\{v_1, \ldots, v_n\}$.

(b) Consider the vector space $\mathbb{P}_3$ consisting of polynomials with degree 3 or less, and the subset

$$S = \{x-1,\ 1+x^2,\ x-x^3,\ x^2+x^3\} \subset \mathbb{P}_3.$$

Does S span $\mathbb{P}_3$? [*15 points*]

$\dim \mathbb{P}_3 = 4$, and S has 4 elements, so S spans $\mathbb{P}_3$ iff it is linearly independent. Write $f_1(x) = x-1$, $f_2(x) = 1+x^2$, $f_3(x) = x-x^3$, $f_4(x) = x^2+x^3$.

Then $(f_1+f_2-f_3-f_4)(x)$

$$= (x-1)+(1+x^2)-(x-x^3)-(x^2+x^3) = 0$$

$\therefore \{f_1, f_2, f_3, f_4\}$ is dependent and hence it does <u>not</u> span $\mathbb{P}_3$.

ALTERNATE PF: If $g \in \operatorname{span} S$, then

$$g(x) = (af_1+bf_2+cf_3+df_4)(x)$$
$$= a(x-1)+b(1+x^2)+c(x-x^3)+d(x^2+x^3)$$
$$= (d-c)x^3+(b+d)x^2+(a+c)x+(b-a)$$

and so writing $g(x) = k_0+k_1x+k_2x^2+k_3x^3$ we have $k_0+k_1 = b+c = k_2-k_3$ $\therefore$ $k_2 = k_0+k_1+k_3$.

This fails for many $g \in \mathbb{P}_3$, including $g(x) = x^2$.

4. Determine whether each of the following maps is linear, with proof.

(a) $T\colon C(\mathbb{R}) \to C(\mathbb{R})$ given by $(Tf)(x) = \int_0^x t(f(t))^2\,dt$. *[10 points]*

T is <u>not</u> linear. Notice that for $c \in \mathbb{R}$ we have

$$(T(cf))(x) = \int_0^x t\cdot(cf(t))^2\,dt = \int_0^x t\,c^2 f(t)^2\,dt$$

$$= c^2 (Tf)(x), \quad \text{so} \quad T(cf) = c^2(Tf).$$

In particular, taking $f(x) = x$ and $c = 2$ gives

$$T(2f) = 4T(f) \neq 2T(f), \quad \text{since}$$

$$(T(f))(x) = \int_0^x t^3\,dt = \frac{1}{4}x^4 \not\equiv 0.$$

(b) $T\colon C(\mathbb{R}) \to C(\mathbb{R})$ given by $(Tf)(x) = \int_0^x t^2 f(t)\,dt$. *[10 points]*

T <u>is</u> linear. Given $f, g \in C(\mathbb{R})$ and $c \in \mathbb{R}$,

we have

$$(T(cf+g))(x) = \int_0^x t^2\cdot(cf+g)(t)\,dt$$

$$= \int_0^x t^2\big(cf(t) + g(t)\big)\,dt$$

$$= c\int_0^x t^2 f(t)\,dt + \int_0^x t^2 g(t)\,dt$$

$$= c(Tf)(x) + (Tg)(x)$$

$$\therefore \quad T(cf+g) = c(Tf) + Tg.$$

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