Math 4377/6308 Advanced Linear Algebra I Fall 2013

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MIDTERM TEST #2

Wednesday, October 30, 2013

You must give complete justification for all answers in order to receive full credit.

Name: SOLUTIONS

	Points	Possible
Problem 1		/20
Problem 2		/35
Problem 3		/20
Problem 4		/25
Total		/100

1. Let V be a vector space over a field K, and let $T: V \to V$ be linear.

(a) Define what it means for v to be an eigenvector of T. [*5 points*]

$v \in V$ is an eigenvector of T if $v \neq \vec{0}$ and $\exists\ \lambda \in K$ s.t. $Tv = \lambda v$.

(b) Define what it means for T to be nilpotent. [*5 points*]

T is nilpotent if $\exists\ k \in \mathbb{N}$ s.t. $T^k = 0$.

(c) Give a 3×3 matrix A such that $A^2 \neq \mathbf{0}$ but $A^3 = \mathbf{0}$. [*5 points*]

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = 0$$

(d) What is the definition of a permutation on n symbols? [*5 points*]

A permutation on n symbols is a 1-1 and onto function $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (2x - 3y, 4x - 6y)$.

(a) Let $\alpha = \{\mathrm{e}_1, \mathrm{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and find the 2×2 matrix $[T]_\alpha$ that represents T relative to α. *[5 points]*

$$Te_1 = T(1,0) = (2,4) = 2e_1 + 4e_2 \Rightarrow [Te_1]_\alpha = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$Te_2 = T(0,1) = (-3,-6) = -3e_1 - 6e_2 \Rightarrow [Te_2]_\alpha = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$$

$$\therefore \quad [T]_\alpha = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$$

(b) Let $\beta = \{\binom{3}{2}, \binom{4}{3}\}$. Write down the change-of-coordinates matrix I_β^α that transforms β-coordinates to α-coordinates. *[5 points]*

$$\beta = \{v_1, v_2\}, \quad [v_1]_\alpha = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } [v_2]_\alpha = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad I_\beta^\alpha = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

(c) Find the coordinate representations $[\mathrm{e}_1]_\beta$ and $[\mathrm{e}_2]_\beta$; also find the change-of-coordinates matrix I_α^β. *[10 points]*

$$I_\alpha^\beta = \left(I_\beta^\alpha\right)^{-1}, \quad \text{so row reduce}$$

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 2 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & 1 & -2 & 3 \end{array}\right] \quad \therefore \quad I_\alpha^\beta = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$[e_1]_\beta = \text{first column of } I_\alpha^\beta = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$[e_2]_\beta = \text{second } \text{'' } \text{'' } \text{'' } = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

(d) Use the change-of-coordinates matrices from the previous part to find $[T]_\beta$. [*10 points*]

$$
\begin{tikzcd}
\alpha\text{-coords} & K^n \arrow[r, "{[T]_\alpha}"] & K^n \arrow[d, "I_\alpha^\beta"] \\
\beta\text{-coords} & K^n \arrow[u, "I_\beta^\alpha"] \arrow[r, "{[T]_\beta}"] & K^n
\end{tikzcd}
$$

$$
\begin{aligned}
[T]_\beta &= I_\alpha^\beta [T]_\alpha I_\beta^\alpha \\
&= \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 5 \\ 0 & -4 \end{bmatrix}
\end{aligned}
$$

(e) Explain why the first column of $[T]_\beta$ is what it is. [*5 points*]

The first column of $[T]_\beta$ is 0 because

$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in N_T$.

3. (a) Let $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$. Given that -1 and 2 are eigenvalues of A, find eigenvectors corresponding to these eigenvalues. [*10 points*]

$\boxed{\lambda = -1}$ eigvecs are in $N_{A-\lambda I}$:

$\cancel{A} - \lambda I = A + I = \begin{bmatrix} 6 & 3 \\ -6 & -3 \end{bmatrix}$

This row reduces to $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is in the nullspace & is an eigvec for -1.

$\boxed{\lambda = 2}$ $A - \lambda I = A - 2I = \begin{bmatrix} 3 & 3 \\ -6 & -6 \end{bmatrix}$

row reduces to $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,

so $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in N_{A-2I}$ $\therefore$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigvec for $\lambda = 2$.

(b) Compute the determinant of the 3×3 matrix $B = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & -2 \\ 4 & 1 & 0 \end{pmatrix}$, **using the method of summing over permutations**. Show which terms correspond to which permutations. [*10 points*]

$$\begin{bmatrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{bmatrix} \quad + \quad 1\cdot 2\cdot 0 = 0 \longrightarrow 0$$

$$\begin{bmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{bmatrix} \quad - \quad 1\cdot 1\cdot(-2) = -2 \longrightarrow +2$$

$$\begin{bmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{bmatrix} \quad - \quad 0\cdot 3\cdot 0 = 0 \longrightarrow 0$$

$$\begin{bmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{bmatrix} \quad + \quad 0\cdot 1\cdot(-1) = 0 \longrightarrow 0$$

$$\begin{bmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{bmatrix} \quad + \quad 4\cdot 3\cdot(-2) = -24 \longrightarrow -24$$

$$\begin{bmatrix} & & \bullet \\ & \bullet & \\ \bullet & & \end{bmatrix} \quad - \quad 4\cdot 2\cdot -1 = -8 \longrightarrow +8$$

Summing these gives

$$\boxed{\det B = -14}$$

4. (a) Given a vector space V and a linear operator $T \in \mathbb{L}(V)$, prove that $T^2 = \mathbf{0}$ if and only if $R_T \subset N_T$. [*10 points*]

($\Rightarrow$) If $T^2 = 0$ and $y \in R_T$, then $\exists\ x \in V$ s.t. $Tx = y$, and so

$Ty = T(Tx) = T^2 x = \vec{0}$ $\therefore$ $y \in N_T$.

Thus $R_T \subset N_T$.

($\Leftarrow$) If $R_T \subset N_T$, then $\forall\ x \in V$ we have $Tx \in N_T$ $\therefore$ $T(Tx) = \vec{0}$.

Thus $T^2 x = T(Tx) = \vec{0}$, and since this holds $\forall\ x \in V$, we have $T^2 = 0$.

(b) Suppose $\dim(V) = 2$ and $T \in \mathbb{L}(V)$ is such that $T \neq \mathbf{0}$ but $T^2 = \mathbf{0}$. Choose any $v_1 \in V \setminus N_T$, and let $v_2 = Tv_1$. Show that $\beta = \{v_1, v_2\}$ is a basis for V, and determine $[T]_\beta$. *[15 points]*

Suppose $v_2 = \lambda v_1$ for some $\lambda \in K$. Then

$Tv_1 = \lambda v_1 \Rightarrow T^2 v_1 = \lambda^2 v_1$

Because $T^2 = 0$ we get $\lambda^2 v_1 = 0$, so $\lambda = 0$ or $v_1 = \vec{0}$. But if $v_1 = \vec{0}$ or $\lambda = 0$ then $v_2 = \lambda v_1 = \vec{0}$, hence $v_1 \in N_T$, contradicting the assumption.

We conclude that v_2 is not a scalar multiple of v_1, and because $v_1 \neq \vec{0}$, this implies that $\beta = \{v_1, v_2\}$ is linearly independent.

Because $\dim V = 2$, this implies that β is a basis.

Now $[Tv_1]_\beta = [v_2]_\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $[Tv_2]_\beta = [\vec{0}]_\beta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore [T]_\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

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