N∞M∞T Lecture Course
Canisius College
Noncommutative functional analysis

David P. Blecher

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Chapter 1

Preliminaries: Matrices = operators

1.1 Introduction

Functional analysis is one of the big fields in mathematics. It was developed throughout the 20th century, and has several major strands. Some of the biggest are:

"Normed vector spaces"

"Operator theory"

"Operator algebras"

We'll talk about these in more detail later, but let me give a micro-summary. Normed (vector) spaces were developed most notably by the mathematician Banach, who not very subtly called them (B)-spaces. They form a very general framework and tools to attack a wide range of problems: in fact all a normed (vector) space is, is a vector space X on which is defined a measure of the 'length' of each 'vector' (element of X). They have a huge theory. Operator theory and operator algebras grew partly out of the beginnings of the subject of quantum mechanics. In operator theory, you prove important things about 'linear functions' (also known as operators) $T : X \to X$, where X is a normed space (indeed usually a Hilbert space (defined below). Such operators can be thought of as matrices, as we will explain soon. Operator algebras are certain collections of operators, and they can loosely be thought of as 'noncommutative number fields'. They fall beautifully within the trend in mathematics towards the 'noncommutative', linked to discovery in quantum physics that we live in a 'noncommutative world'. You can study a lot of 'noncommutative mathematics' in terms of operator algebras.

The three topics above are functional analysis. However, strangely, in the course of the decades, these subjects began to diverge more and more. Thus, if you look at a basic text on normed space theory, and a basic text on operator algebras, there is VERY little overlap. Recently a theory has developed which builds a big bridge between the two. You can call the objects in the new theory 'noncommutative normed spaces' or 'matrix normed spaces, or 'operator spaces'. They are, in many ways, more suited to solving problems from 'Operator algebras' and 'noncommutative mathematics'. In this course, you will get a taste of some of the beautiful ideas in and around this very active and exciting new research area. We will need to lay a lot of background, but please be patient as we go step-by-step, starting from things you know, and then getting to graduate level material!

The title of this series is **Noncommutative functional analysis**. Let us first look at the first word here: **noncommutative**. Remember, commutativity means:

$$a\,b \;=\; b\,a\,.$$

Thus noncommutativity means:

$$a\,b \;\neq\; b\,a\,.$$

You have probably seen the difference already in calculus classes. For example consider the function $\sin(x^2)$. This is not $\sin^2(x)$. Also, $\sin(2x) \neq 2\sin(x)$, and $\ln(\sin x) \neq \sin(\ln x)$. The *order of operations* usually matters in mathematics. On the other hand,

$$\sin(x)\;\ln(x) \;=\; \ln(x)\;\sin(x)\,.$$

Why does this happen? Why the difference?

Answer: the difference lies in the product you are using. When we do $\ln(\sin x)$ we mean the 'composition product' $\ln \circ \sin$. When we do $\ln x \;\; \sin x$ we are doing the 'pointwise product' of functions. The 'composition product' $f \circ g$ is usually noncommutative. The 'pointwise product' $f \cdot g$ defined by $(f \cdot g)(x) \;=\; f(x)g(x)$ is commutative, at least if f and g are scalar-valued functions like ln and sin.

We will introduce our language as we go along. By a *scalar*, we mean either a real or a complex number. We will write $\mathbb{F}$ to denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A *scalar valued function* on a set A is a function from $f : A \to \mathbb{F}$. The numbers $f(x)$, for x in the domain of f, are called the *values* of f.

Moral: The commutativity we saw above for the 'pointwise product' $f \cdot g$, is explained by the commutativity in $\mathbb{R}$ or $\mathbb{C}$, i.e. by the commutativity of scalars.

In the beginning ... (of functional analysis) ... was the MATRIX ...

Another example of noncommutativity is matrix multiplication: in general

$$AB \neq \; BA$$

for matrices. Matrices are the *noncommutative version* of scalars.

Because matrices play such an important role, lets remind you about some basic definitions:

- An $m \times n$ matrix A is written as

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Here a_{ij} is the *entry* in the ith row and jth column of A. We also write $A = [a_{ij}]$. We write M_{mn} for the vector space of $m \times n$ matrices with scalar entries; and $M_n = M_{nn}$, the *square matrices*. If we want to be specific as to whether the entries are real or complex scalars, we may write $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$. For example, the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \\ -1 & 7 \end{bmatrix}$$

is in $M_{3,2}(\mathbb{R})$, and $a_{2,2} = 3$ (the 2-2 entry of A). We add matrices A and B by the rule $A + B = [a_{ij} + b_{ij}]$; that is we add *entrywise*. Also $\lambda A = [\lambda a_{ij}]$, for a scalar λ. These operations make $M_{n,m}$ a vector space.

- The *transpose* A^t of a matrix A:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}.$$

In other notation, $[a_{ij}]^t = [a_{ji}]$.

- A 'diagonal matrix':

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}.$$

That is, the matrix is all zeroes, except on the *main diagonal*. Note that two diagonal matrices commute. You can think of diagonal matrices as commutative objects living in the noncommutative world (namely, M_n). The identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

and scalar multiples of it, commutes with everything. Indeed I_n plays the role the number 1 plays in the scalar field.

- 'Orthogonal matrices' play the role the numbers ± 1 play in the scalar field. The 'unitary matrices' play the role the complex numbers of absolute value 1 play in the scalar field. Reminder: an orthogonal matrix is a matrix A such that $A^t A = AA^t = I_n$.

A *unitary* matrix is a matrix A such that $A^*A = AA^* = I_n$; thus in this case $A^{-1} = A^*$ (Why?). Here A^* is the 'conjugate transpose' defined by

$$[a_{ij}]^* = [\overline{a_{ji}}].$$

(So we take the transpose, and also replace each entry by its complex conjugate). Also, these matrices can be regarded as giving a nice 'change of orthonormal basis'. We'll say more about this later.

- In the noncommutative world, we view matrices as a noncommutative version of scalar-valued functions. The *spectral values* of an $n \times n$ matrix A, are its *eigenvalues*, that is, the complex numbers λ such that $\lambda I_n - A$ is not invertible.

- One of the fundamental theorems from undergraduate mathematics is the fact that any *selfadjoint* matrix A can be *diagonalized*. We say that A is selfadjoint if $A = A^*$. (Note that if A is a real matrix, then this property is also called being 'symmetric', it means the matrix is 'symmetric' about its 'main diagonal'). To say that A can be *diagonalized* means that there is a matrix U with $A = U^{-1}DU$ where D is a diagonal matrix. In fact, the numbers on the diagonal of D are exactly the eigenvalues of A, which is very nice!! Also, U can be chosen to be a unitary matrix, which is very important Also, U can be chosen to be a unitary matrix, which is very important.

At first sight perhaps this handout may look off-putting; however much of it you know already in some fashion. Persevere with it, and quickly you will be more conversant with the terms and ideas. Read the material below in the given order—a lot of items use notation and language and ideas from previous items.

1.2 A formula from linear algebra

The subject of these lectures is somewhere on the border of analysis and algebra and ... .

This lecture is centered around one basic formula from linear algebra, which we will stretch in various directions, seeing what we can learn about variations on this formula, and about mathematics in general. If you stay with me, you will learn a lot of things that will be very useful in other math classes, such as algebra, real analysis, matrix theory,..., and also in some physics classes.

One of the most important, and basic, facts in linear algebra is the Equation

$$M_n \cong Lin(\mathbb{R}^n). \tag{1}$$

We may summarize it as the important Principle:

$$\text{Matrices} = \text{Operators}$$

Indeed Equation (1) says that the space $M_n(\mathbb{R})$ of $n \times n$ matrices with real scalar entries 'is the same as' the set of linear functions (also known as 'operators') from $\mathbb{R}^n$ to itself. Similarly,

$M_n \cong Lin(\mathbb{C}^n)$, if M_n now are the matrices with complex scalar entries. In fact, this version is 'better', because complex numbers are more natural when working with matrices, because even real matrices can have complex eigenvalues. In explaining what we are doing, it doesnt really matter if you use $\mathbb{R}$ or $\mathbb{C}$, so for simplicity we will mainly talk about the $\mathbb{R}$ case.

And when we say 'M_n is the same as $Lin(\mathbb{R}^n)$', we mean that the two sides of Equation (1) are 'isomorphic'. What does 'isomorphic' mean here? To understand this, we need to look at the algebraic structure of the two sides of equation (1). Note that M_n is a vector space (so it has a $+$, and a scalar multiplication). Also, M_n has a product, the product of matrices. Similarly, $Lin(\mathbb{R}^n)$ is a vector space (the $+$ is the usual sum of functions, and the scalar multiplication is what you would expect: e.g. $(3T)(x) = 3T(x)$ for $T \in Lin(\mathbb{R}^n), x \in \mathbb{R}^n$). Similarly, $Lin(\mathbb{R}^n)$ has a product, the 'composition product' mentioned in the last section. So to say that $M_n \cong Lin(\mathbb{R}^n)$ ought to mean that there exists a function $f : M_n \to Lin(\mathbb{R}^n)$ which is one-to-one, onto, is linear, and 'preserves products' (i.e. $f(ab) = f(a)f(b)$ for all $a, b \in M_n$).

In mathematics, we quickly try to name the structures we see. Names are very important, without them one quickly loses track of what we are talking about, one loses track of the structures that are around, and then one cannot see the wood for the trees. Let us give some names to some of the structures we saw above. You will have to know what these names mean later: A function which is linear and which 'preserves products' is called a *homomorphism* or an *algebra homomorphism*; vector spaces which have a product are called *algebras*.

You may remember what the function f has to be, in order to prove Equation (1). It is the function f that takes a matrix $a \in M_n$ to the 'operator' L_a on $\mathbb{R}^n$, where

$$L_a(x) = ax \ , \qquad \forall x \in \mathbb{R}^n.$$

In linear algebra class you proved that indeed this function f is an algebra homomorphism, and is one-to-one and onto. Because this Equation (1) will be so important, let me remind you of the proof. Suppose that $a, b \in M_n$ and λ is a scalar. Then:

- $(L_a + L_b)(x) = ax + bx = L_{a+b}(x)$ for any x. Thus $f(a) + f(b) = f(a + b)$.
- $(\lambda L_a)(x) = \lambda L_a(x) = \lambda ax = L_{\lambda a}(x)$ for any x. Thus $f(\lambda a) = \lambda f(a)$. We have now proved that f is *linear*.
- $(L_a L_b)(x) = L_a(L_b(x)) = abx = L_{ab}(x)$ for any x. Thus $f(a)f(b) = f(ab)$. We have now proved that f is a *homomorphism*.
- If $f(a) = 0$ then $ax = L_a(x) = 0$ for all x. Remember that if $\{e_1, \cdots, e_n\}$ is the *standard basis* of $\mathbb{R}^n$, then ae_k is the k'th column of a, for every k. So every column of a is 0, so $a = 0$. We have now proved that f is one-to-one.
- Finally, to see that f is onto, if $T \in Lin(\mathbb{R}^n)$, we let a be the matrix whose k'th column is $T(e_k)$, where e_k is as above. As we remarked above, $ae_k = L_a(e_k)$ is the k'th column

of a. So $L_a(e_k) = T(e_k)$. Since both L_a and T are linear,

$$L_a(\sum_k x_k e_k) = \sum_k x_k L_a(e_k) = \sum_k x_k T(e_k) = T(\sum_k x_k e_k), \qquad \forall \text{ scalar } x_1, x_2, \cdots, x_n.$$

Thus $L_a(x) = T(x)$ for all $x \in \mathbb{R}^n$, so that $T = L_a = f(a)$. So f is onto.

1.3 The adjoint

We are going to look at several variants of the Equation (1). First, I want to look at another, more hidden, structure that both M_n and $Lin(\mathbb{R}^n)$ have. Each has a natural structure on it which we can call an 'adjoint' or 'involution'. In $M_n(\mathbb{R})$ this is just the transpose. However, if we are working in the $\mathbb{C}$ field rather than $\mathbb{R}$, it is better to use the 'conjugate transpose' defined by

$$[a_{ij}]^* \;=\; [\overline{a_{ji}}].$$

(So we take the transpose, and also replace each entry by its complex conjugate). Notice that $(A^*)^* = A$. Also we will want our involutions to have a few other 'obvious properties', like:

$$(A+B)^* = A^* + B^* \;, \qquad (AB)^* = B^* A^*.$$

These are pretty easy to see in $M_n(\mathbb{R})$, where $*$ is the transpose, but its also easy to check for the 'conjugate transpose'.

What is the 'adjoint' or involution on $Lin(\mathbb{R}^n)$? To explain this, we need to remember that $\mathbb{R}^n$ (and $\mathbb{C}^n$) has a 'dot product' (also known as an 'inner product' or 'scalar product'). We write this as $\langle x, y\rangle$. The formula is:

$$\langle x, y\rangle \;=\; \sum_k x_k \overline{y_k},$$

where x_k is the kth coordinate of x and y_k is the kth coordinate of y. If $T \in Lin(\mathbb{R}^n)$ or $Lin(\mathbb{C}^n)$ then we will show in a few minutes, Claim 1: there exists a operator S such that

$$\langle T(x), y\rangle \;=\; \langle x, S(y)\rangle.$$

This operator S is written as T^* (it is uniquely determined by the last formula, as you can check as an exercise), and it is called the 'adjoint' or 'involution' of T.

Thus both sides of the Equation (1) have an 'adjoint' $*$. They may thus be called '$*$-algebras'. Claim 2: the isomorphism f that proved Equation (1) above 'preserves this new structure too'. That is, $f(a^*) = f(a)^*$ for all matrices a. We say that f is a $*$-*homomorphism*, indeed a $*$-*isomorphism*.

We can prove both Claim 1 and Claim 2 in one shot, using a basic property of the transpose ^{t} of matrices, namely that $(AB)^t = B^t A^t$. Suppose that $a \in M_n$ and $x, y \in \mathbb{R}^n$. Then $\langle x, y\rangle = y^t x$. Thus

$$\langle f(a)x, y\rangle = \langle ax, y\rangle = y^t a x = (a^t y)^t x = \langle x, a^t y\rangle = \langle x, f(a^t) y\rangle.$$

Since f is onto, this proves Claim 1. It then follows that $f(a)^* = f(a^t)$, which is Claim 2. A similar proof works in the complex case.

Exercise: Show that $(T^*)^* = T$ and $(ST)^* = T^*S^*$.

The main point: we now see that Equation (1) is true in an even stronger sense, the two sides are 'the same' not only as algebras, but also as $*$-algebras.

Why is this so good?

Answer: it allows the introduction of the important notions of 'positivity' and 'selfadjointness' into Equation (1). Lets discuss 'selfadjointness' first. We say that x is selfadjoint if $x = x^*$. (Note that if x is a real matrix, then this property is also called being 'symmetric', it means the matrix is 'symmetric' about its 'main diagonal'.) The fact that f is a $*$-homomorphism implies that if x is selfadjoint then $f(x)^* = f(x^*) = f(x)$, so that $f(x)$ is selfadjoint. So f takes selfadjoint things to selfadjoint things. Selfadjointness is incredibly important, for example in quantum physics. It is the noncommutative analogue of being 'real valued'. So we now have a version of Equation (1) that is better for quantum physics, for example.

Next lets say something about positivity of matrices, again incredibly important in quantum physics and elsewhere. To understand positivity, lets first look at positivity of scalars. A scalar $\lambda \in \mathbb{C}$ satisfies $\lambda \geq 0$ iff $\lambda = \bar{z}z = |z|^2$ for a number $z \in \mathbb{C}$. Next lets look at positivity of scalar-valued functions. For a scalar-valued function $f : K \to \mathbb{C}$ the following are all equivalent to saying that $f \geq 0$:

(a) the values of f, $f(x)$, are all ≥ 0.

(b) there is another scalar-valued function g such that $f = \bar{g}g$. That is, $f(x) = \overline{g(x)}g(x) = |g(x)|^2$ for all $x \in K$.

(c) ...

Now let us turn to matrices: In the noncommutative world, we view matrices as a noncommutative version of scalar-valued functions. The *spectral values* of an $n \times n$ matrix A, are its *eigenvalues*, that is, the complex numbers λ such that $\lambda I_n - A$ is not invertible. If a matrix is selfadjoint, you can show that its eigenvalues are all real numbers. It turns out that the following are all equivalent, for an $n \times n$ matrix A:

(a) A is selfadjoint and its spectral values are all ≥ 0.

(b) there is another matrix $B \in M_n$ such that $A = B^*B$.

(c) ...

A matrix with these properties will be called positive, and we write $A \geq 0$. Note that this is different to saying that all the 'entries' of A are ≥ 0.

What does positivity mean in $Lin(\mathbb{R}^n)$ or $Lin(\mathbb{C}^n)$? You can define it, for example, like condition (b) above, namely $T \geq 0$ iff $T = R^*R$ for some $R \in Lin(\mathbb{R}^n)$.

Look at condition (b) above. If f is our $*$-isomomorphism above (the one giving Equation (1), then

$$f(B^*B) \;=\; f(B^*)f(B) \;=\; f(B)^*f(B) \;\geq\; 0.$$

Thus the isomorphism in Equation (1) takes positives to positives; which is very important in science.

1.4 Add some analysis...

Remember that in this lecture, we are going to look at several variants of the Equation (1). Let us now add a little analysis to the mix. In analysis, one is often interested in 'size' and 'distance'. These are usually measured by *norms*. It turns out that the three spaces $\mathbb{R}^n, M_n, Lin(\mathbb{R}^n)$ have natural 'norms'. The norm we will always use on $\mathbb{R}^n$ is called the *Euclidean norm* or *2-norm*:

$$\left\| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\|_2 \;=\; \sqrt{\sum_{k=1}^{n} |x_k|^2}\;.$$

Before we discuss the natural norms on M_n and $Lin(\mathbb{R}^n)$, lets talk a little more about general norms.

(a) Norms

A *norm* on a vector space V is a function $\|\cdot\| : V \to [0,\infty)$ satisfying the following properties:

(i) $\|x\| \geq 0$ for all $x \in V$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{F}$ and $x \in V$,

(iii) (Triangle inequality) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$,

(iv) $\|x\| = 0$ implies that $x = 0$.

- On $\mathbb{R}^n$ (or $\mathbb{C}^n$) we will only use the Euclidean norm, i.e. the norm $\|\vec{v}\|_2 = \sqrt{\sum_k |v_k|^2}$, for $\vec{v} \in \mathbb{F}^n$. You will probably have seen the proof in one of your classes that it is a norm (the hardest part is checking the triangle inequality).
- You should think of the quantity $\|x-y\|$ as the distance between x and y.
- If $\|\cdot\|$ is a norm on a vector space V, then we say that $(V, \|\cdot\|)$ is a *normed vector space* (or normed linear space, or *normed space*). A normed space X is called *complete* if every Cauchy sequence in X converges to a point in X. A *Banach space* is a normed vector space which is complete in this sense. In this course we will not worry too much about convergence of Cauchy sequences; its not hard, but its a technicality that obscures the really key points.

- Let X be a normed space. We write $\text{Ball}(X)$ for the set $\{x \in X : \|x\| \leq 1\}$.

  Note that if x is any vector in a normed space, $x \neq 0$, then by 'scaling' one gets a vector of norm 1. That is, $x/\|x\|$ is a vector of norm 1. we also call this 'normalizing', and $x/\|x\|$ is called a normalized vector. Note $x/\|x\|$ is in $\text{Ball}(X)$.

- A linear subspace W of a normed space X is of course also a normed space, with the inherited norm. We will often simply say 'subspace' for a linear subspace.

  If $T : X \to Z$, and if $W \subset X$ is a subspace, then we write $T_{|W}$ for the function from W to Z obtained by restricting T to W. If T is linear then of course so is $T_{|W}$.

- We are wanting to see that there is a natural norm on $Lin(\mathbb{R}^n)$. In fact, this fits into the following general theory:

- For a linear operator $T : X \to Y$, we define the norm of T, namely $\|T\|$, to be the least constant M such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$. If $\|T\| < \infty$ then we say that T is *bounded*. This is always the case if X is finite dimensional (we omit the proof, which is not hard). In particular, we have

$$\|T(x)\| \leq \|T\|\|x\|, \qquad \forall x \in X \ .$$

  Other useful formulae for $\|T\|$ are

$$\|T\| = \sup\{\|T(x)\| : x \in \text{Ball}(X)\}$$

$$\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| < 1\}$$
$$\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| = 1\}$$
$$\|T\| = \sup\{\frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0\}$$

  —these numbers turn out to be all the same (Exercise with hint: multiply x by various positive scalars, and use the fact that T is linear).

  It also turns out that T is bounded if and only if T is *continuous*, which is nice. We won't particularly use this, so we omit the proof (which is not hard).

  We write $B(X, Y)$ for the set of bounded linear operators from X to Y, when X and Y are normed spaces. As we said above, if X is finite dimensional then $Lin(X, Y) = B(X, Y)$ as sets of functions. It is easy to check that $B(X, Y)$ is also a normed space with the norm $\|T\|$ above. Thus for example $\|S+T\| \leq \|S\|+\|T\|$, for $S, T \in B(X, Y)$. (Exercise: check it).

- A special case of particular interest is when Y is just the scalars; we write X^* for $B(X, \mathbb{R})$, and call this space the *dual space* of X. The functions in X^* are called *functionals*. This explains the second word in the title of this course "Noncommutative functional analysis".

- Another special case of particular interest is when $X = Y$. In this case we write $B(X, Y)$ as $B(X)$. In addition to $B(X)$ being a normed space, it is an algebra. That is, the 'composition product' of bounded linear operators is a bounded linear operator. Indeed, it also has the following nice property:

$$\|ST\| \leq \|S\| \, \|T\|, \qquad \forall S, T \in B(X).$$

To see that what we are talking about is not hard, lets take the time to prove this:

So suppose that $S, T \in B(X)$. Then $S \circ T$ is clearly linear. For example, for $x, y \in X$ the quantity $(S \circ T)(x + y)$ equals

$$S(T(x + y)) = S(T(x) + T(y)) = S(T(x)) + S(T(y)) = (S \circ T)(x) + (S \circ T)(y),$$

for $x, y \in X$. Also,

$$\|(S \circ T)(x)\| = \|S(T(x))\| \leq \|S\|\|T(x)\| \leq \|S\|\|T\|\|x\|.$$

Hence $\|ST\| \leq \|S\| \, \|T\|$.

- We have now explained what is the natural norm on $Lin(\mathbb{R}^n)$. It is the norm above, for example,

$$\|T\| = \sup\{\|T(x)\|_2 : x \in \mathbb{R}^n, \|x\|_2 \leq 1\}.$$

(And similarly for the natural norm on $Lin(\mathbb{C}^n)$).

(b) The norm of a matrix

- In the noncommutative world, we view matrices as a noncommutative version of scalar-valued functions. Remember that the 'values' of a scalar-valued function $f : K \to \mathbb{C}$ say, are the complex numbers $f(x)$, for $x \in K$. The *spectral values* of an $n \times n$ matrix A, are its *eigenvalues*, that is, the complex numbers λ such that $\lambda I_n - A$ is not invertible.

- Before we come to the norm of a matrix, let us talk about the natural norm of a scalar-valued function $f : K \to \mathbb{C}$. If $f(x) \geq 0$ for all x, we just define the norm of f to be

$$\|f\|_\infty = \sup\{f(x) : x \in K\}.$$

In the general case, we can define

$$\|f\|_\infty = \sup\{|f(x)| : x \in K\} = \sup\{|f(x)|^2 : x \in K\}^{\frac{1}{2}}.$$

This shows how to define the norm of a matrix. If A is a matrix which is 'positive' (i.e. $A \geq 0$), we define $\|A\|$ to be the largest eigenvalue of A. If A is not positive we define $\|A\|$ to be the square root of the largest eigenvalue of A^*A. (This is the same as the largest eigenvalue of $|A|$, but I dont want to take the time to define $|A|$ for a matrix A.) The fact that this does satisfy the properties of a norm is easily seen, for example, from what comes next. Notice that $\|A\|^2 = \|A^*A\|$, which is just a version, for matrices of the formula $|z|^2 = \bar{z}z$ valid for complex numbers. So M_n is a noncommutative version of the complex number field.

(c) The 'analysis version' of Equation (1)

Remember Equation (1):

$$M_n \;\cong\; Lin(\mathbb{R}^n).$$

We have seen that both M_n and $Lin(\mathbb{R}^n)$ have natural norms. Fortunately, the isomorphism $f : M_n \to Lin(\mathbb{R}^n)$ also preserves these norms!! That is, $\|f(a)\| = \|a\|$, for all $a \in M_n$. Exercise: Prove this (Hint: First rephrase this as the statement that the largest eigenvalue of a^*a equals

$$\sup\{\|ax\|^2 : x \in \mathrm{Ball}(\mathbb{R}^n)\} = \sup\{\langle ax, ax\rangle : x \in \mathrm{Ball}(\mathbb{R}^n)\} = \sup\{\langle a^*ax, x\rangle : x \in \mathrm{Ball}(\mathbb{R}^n)\}.$$

Prove the latter statement, by first 'diagonalizing' a^*a.)

Let us use some more precise language. First we remark that a linear operator T between normed spaces with $\|T\| \leq 1$ is called a *contraction*. A linear operator $T : X \to Y$ with $\|T(x)\| = \|x\|$ for all $x \in X$, is called an *isometry*. Note that if T is an isometry then if $T(x) = 0$ then $\|x\| = \|T(x)\| = 0$ so that $x = 0$. Thus T is 1-1. A linear isometry $T : X \to Y$ which is onto is called an *isometric isomorphism*. These are very important. If such an isometric isomorphism exists we say that X and Y are *isometrically isomorphic*, and write $X \cong Y$ *isometrically*. In this case we often think of X and Y as being essentially the same. Indeed because T respects all the structure (the vector space structure and the norm), whatever is true about X as a normed space will be true about Y.

Thus we can summarize most of what we have done in this first lecture by saying that the function $f : M_n \to B(\mathbb{R}^n)$ is an *isometric $*$-isomorphism*. The key point now, is that the two sides of Equation (1) are also 'equal' in the sense of 'analysis'. The two sides are 'equal' as normed spaces.

Thus the norm of a matrix a is given by the formula:

$$\|a\| = \sup\{\|ax\|_2 : x \in \mathrm{Ball}(\mathbb{R}^n)\}.$$

This expression is called the *operator norm of the matrix*.

There is another important formula for the operator norm of a matrix. It is:

$$\|[a_{ij}]\| \;=\; \sup\,\Big\{\,\Big|\sum_{ij} a_{ij} z_j \overline{w_i}\Big| \;:\; z = [z_j],\;\; w = [w_i] \in \mathrm{Ball}(\mathbb{R}^n)\Big\}.$$

To prove this, we will use the fact that for any vector $z \in \mathbb{R}^n$, we have

$$\|z\|_2 \;=\; \sup\{|\langle z, y\rangle| : y \in \mathrm{Ball}(\mathbb{R}^n)\}.$$

To prove the last formula, note that the right side is less than or equal to the left side by the well-known Cauchy-Schwarz inequality $|\langle z, y\rangle| \leq \|z\|_2\|y\|_2$. (Note that the Cauchy-Schwarz inequality says that the absolute value of the dot product of two vectors, is $\leq$ the product of the lengths of the two vectors. You may have seen a proof of it, or of a form of it–it is quite easy to prove). On the other hand, if $y = z/\|z\|_2$ then $\|y\|_2 = 1$, so that $y \in \mathrm{Ball}(\mathbb{R}^n)$, and

$$|\langle z, y\rangle| = \frac{|\langle z, z\rangle|}{\|z\|_2} = \frac{\|z\|_2^2}{\|z\|_2} = \|z\|_2,$$

which shows that the right side is greater than or equal to the left side. Putting the formula which we have just proved together with the formula in the last paragraph, we have

$$\begin{aligned}\|A\| &= \sup\{\|Ax\|_2 : x \in \mathrm{Ball}(\mathbb{R}^n)\}\\ &= \sup\{\sup\{|\langle Ax, y\rangle| : y \in \mathrm{Ball}(\mathbb{R}^n)\} : x \in \mathrm{Ball}(\mathbb{R}^n)\}\\ &= \sup\{|\langle Ax, y\rangle| : x, y \in \mathrm{Ball}(\mathbb{R}^n)\},\end{aligned}$$

which is the same as the thing we are trying to prove.

1.5 The infinite dimensional version of Equation (1)

Remember Equation (1):

$$M_n \cong Lin(\mathbb{R}^n).$$

What if $n = \infty$ (which is often the most important case in applications)? Is there a version of this which is true? To understand this, we will need infinite dimensional versions of M_n and $\mathbb{R}^n$. Also, we will replace *Lin* by the set of bounded operators discussed earlier.

First, how to generalize Euclidean space $\mathbb{R}^n$ to infinite dimensions? Probably most of you have seen this. There are two main ways to do it, but fortunately they are equivalent. The first way is to work with infinitely long columns of scalars. The Euclidean norm (or *2-norm*) has the same formula, namely

$$\left\| \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \right\|_2 = \sqrt{\sum_{k=1}^{\infty} |x_k|^2}\ .$$

We replace $\mathbb{R}^n$ by the set of infinitely long columns whose 2-norm is a finite number. This set is called ℓ^2 usually, and it can be shown to also be a normed space with the 2-norm. In fact it is more than a normed space, it is what is known as a *Hilbert space*. And this has led us to the second way of generalizing Euclidean space $\mathbb{R}^n$ to infinite dimensions. Before we define Hilbert spaces, we need some more background. Some of you will know all this... . You should have met some of it in linear algebra, perhaps under the name scalar product or dot product:

- An *inner product space* is a vector space H over the field $\mathbb{F}$ (here $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ as usual), with an inner product: that is, a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}$ with the following properties:

  (i) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in H$; and $\langle \alpha x, z\rangle = \alpha\langle x, z\rangle$ for all $x, z \in H$ and scalar α (If these hold we say the function is linear *in the first variable*),

  (ii) $\langle x, x\rangle \geq 0$ for all $x \in H$,

  (iii) $\langle x, x\rangle = 0$ if and only if $x = 0$,

(iv) $\langle x, y\rangle = \langle y, x\rangle$ if the underlying field $\mathbb{F}$ is $\mathbb{R}$, otherwise we insist that $\langle x, y\rangle$ is the complex conjugate of $\langle y, x\rangle$ for all $x, y \in H$.

For such an inner product on H we define $\|x\| = \sqrt{\langle x, x\rangle}$. One can show that this is a norm. This is proved using the Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

for all $x, y \in H$. We omit the easy (three line) proof, which you've probably seen somewhere.

- A *Hilbert space* is an inner product space for which the associated norm $\|x\| = \sqrt{\langle x, x\rangle}$ is a complete norm; i.e. Cauchy sequences converge. In this course we will not worry too much about convergence of Cauchy sequences; its not hard, but its a technicality that obscures the really key points. So if you like, for this course, think of Hilbert spaces and inner product spaces as being the same thing.

- Examples: From linear algebra you should remember that Euclidean space $\mathbb{R}^n$ is an inner product space, with the dot product as the inner product. Similarly, for $\mathbb{C}^n$. On ℓ^2, define $\langle x, y\rangle = \sum_k x_k \overline{y_k}$, where x_k is the kth coordinate of x and y_k is the kth coordinate of y. It is easy to check that this is an inner product! (Exercise: show it). The associated norm, $\sqrt{\langle x, x\rangle} = \sqrt{\sum_k x_k \overline{x_k}} = \sqrt{\sum_k |x_k|^2}$, which is just the 2-norm. So ℓ^2 is a Hilbert space.

- A very important notion is that of *unitary operators* (called unitaries for short). If H and K are two inner product spaces then a linear $U : H \to K$ is unitary if and only if U is invertible, and $U^* = U^{-1}$. This is the same as saying that U is onto, and $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in H$. (You can try this as an exercise). A little harder (not much) is that this is the same as saying that U is isometric and onto. In fact unitaries may be thought of as nothing more than a 'change of orthonormal basis', if you know what that means. The assertions I've just made are often proved in a linear algebra class—see your linear algebra text.

- The theory of Hilbert spaces is an exceptionally pretty and useful part of mathematics. Everything works out so nicely!! For example, even if they are infinite dimensional, their theory is very similar to that of Euclidean n space. Indeed up to unitary isomorphism, there is only one Hilbert space of any given dimension. That is, any Hilbert space of dimension n is unitarily isomorphic to $\mathbb{R}^n$ (or $\mathbb{C}^n$). This follows easily from from the fact which you probably proved in linear algebra that every finite dimensional inner product space has an 'orthonormal basis' (also known as the Gram-Schmidt process, don't worry if you dont know this). Similarly, ℓ^2 is the 'only' Hilbert space of its dimension.

So we now know how to generalize the right side of Equation (1), we can replace $Lin(\mathbb{R}^n)$ by $B(\ell^2)$, or by a general Hilbert space H. This has a natural norm, as we saw earlier. It is also an algebra, since as we checked earlier, $B(X)$ is an algebra for any normed space X. Is it a $*$-algebra? That is, is there a natural 'adjoint' or 'involution' $*$ here? In fact the answer is YES, for reasons almost identical to what we saw in the case of $Lin(\mathbb{R}^n)$. Namely, every $T \in B(H)$ has an involution T^*, defined to be the (unique) operator S such that

$$\langle Tx, y\rangle \;=\; \langle x, Sy\rangle\ , \qquad \forall x, y \in H.$$

We omit the proof that such an S exists. Thus, as before, $B(H)$ is a $*$-algebra for any Hilbert space H; in particular, $B(\ell^2)$ is a $*$-algebra.

Now lets turn to the infinite generalization of M_n, the left side of Equation (1). We can replace $n \times n$ matrices with infinite matrices

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

For such an infinite matrix A, let A_n be the $n \times n$ matrix in the top left corner of A. For example, $A_1 = a_{11}$,

$$A_2 \;=\; \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and so on. We define $\|A\| = \sup\{\|A_n\| : n \in \mathbb{N}\}$. Define M_∞ to be the set of such infinite matrices A such that $\|A\| < \infty$. The promised infinite dimensional generalization of Equation (1) is then:

$$M_\infty \;\cong\; B(\ell^2).$$

Note that M_∞ is a vector space (Exercise: check it!), and it has a product (the matrix product), and an involution $*$ which is defined like the one we studied on M_n. One can show that, just as in the $\mathbb{R}^n$ case, the relation $M_\infty = B(\ell^2)$ is true isometrically, and as $*$-algebras! I will not prove it; some of you may be able to prove it as a (difficult) homework exercise!

Main point: There is a good, and not too difficult, generalization of everything we said in the $\mathbb{R}^n$ case, to infinite dimensions. Again, this means that the isomorphism takes 'selfadjoints' to 'selfadjoints', and 'positives' to 'positives', which is very important, e.g. in quantum mechanics.

Indeed, for any Hilbert space, $B(H)$ is $*$-isomorphic to a space of matrices. Lets prove this. Assuming that H is not pathologically big, it follows by what we said above, that there is a unitary U from H onto $\mathbb{R}^n$ or onto ℓ^2. Lets suppose the latter, for example. Define a function $g : B(H) \to B(\ell^2)$ by $g(T) = UTU^* = UTU^{-1}$. Exercise: g is a one-to-one $*$-homomorphism. It is easy to see that g is onto, indeed it has an inverse, the function $S \mapsto U^{-1}SU$. So $B(H)$ is $*$-isomorphic to $B(\ell^2)$. On the other hand, we saw that $B(\ell^2)$ is $*$-isomorphic to M_∞. Composing these two $*$-isomorphisms, we deduce $B(H) \cong M_\infty$.

Moral: Every operator on a Hilbert space can be viewed as a matrix. Thus again, 'matrices = operators'.

1.6 A final generalization of Equation (1)

We end this Chapter with another major generalization of Equation (1), namely the following principle (also crucial in e.g. quantum mechanics):

A matrix of operators is an operator!

Notice that Equation (1) says: 'a matrix of scalars is an operator'. Since any operator can be viewed as a matrix, another way to state the new principle is that 'a matrix of matrices is again a matrix'. And this is intuitively obvious if you look at the following example:

$$\begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 4 \\ 5 & 5 \end{bmatrix} & \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 2 & 5 & 6 \\ 2 & 3 & 3 & 4 & 7 & 8 \\ 9 & 0 & 3 & 4 & 5 & 4 \\ 1 & 2 & 5 & 5 & 3 & 2 \\ 1 & 0 & 2 & 0 & -1 & 2 \\ 1 & 2 & 3 & 1 & 1 & 3 \end{bmatrix}.$$

Note that you can view this as just 'erasing the inner matrix brackets'. We wish to make this principle more 'mathematical', more 'precise'. Mathematically, the new principle can be phrased more precisely as the algebraic formula:

$$M_n(B(H)) \cong B(H^{(n)}) . \qquad (2)$$

This relation will also be crucial to us later when we discuss noncommutative functional analysis, so I want to explain it in some detail. First, H is a Hilbert space (e.g. Euclidean space, or ℓ^2). What does $M_n(B(H))$ mean? Generally in these talks, if X is any vector space, then $M_n(X)$ means the set of $n \times n$ matrices with entries in X. This is again a vector space if X is a vector space (Exercise: show it). Indeed, it is again an algebra if X is an algebra, its product is the usual way we multiply matrices. Finally, $M_n(X)$ is again a $*$-algebra if X is a $*$-algebra; the 'involution' is given by the formula

$$[x_{ij}]^* = [x_{ji}^*].$$

Thus $M_n(B(H))$ is a $*$-algebra, since we saw earlier that $B(H)$ is a $*$-algebra. Now let us turn to the right side of Equation (2). I must explain $H^{(n)}$. If $H = \mathbb{R}^m$, then $H^{(n)} = \mathbb{R}^{mn}$. More generally, $H^{(n)}$ is defined to be the new inner product space which is $H \oplus H \oplus \cdots \oplus H$ (or if you prefer, $H \times H \times \cdots H$, the Cartesian product of n copies of H). A typical element of $H^{(n)}$ should be regarded as a column

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x_1, x_2, \cdots, x_n \in H.$$

The inner product of two such columns is just:

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} , \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_k \langle x_k, y_k \rangle \ .$$

Notice that if $H = \mathbb{R}$ then $H^{(n)}$ is just $\mathbb{R}^n$ with its usual dot product. If $H = \mathbb{R}^m$, then $H^{(n)}$ is just $\mathbb{R}^{nm}$.

In general, it is easy to see that $H^{(n)}$ is an inner product space, if H is an inner product space. Indeed $H^{(n)}$ is a Hilbert space, if H is a Hilbert space. So now we understand the right side of Equation (2); note that $B(H^{(n)})$ is a $*$-algebra, since $B(H)$ is a $*$-algebra for any Hilbert space H, and hence it is in particular for the Hilbert space $H^{(n)}$.

We can now understand Equation (2), the formula $M_n(B(H)) \cong B(H^{(n)})$, as saying that these two $*$-algebras are $*$-isomorphic. What is the function $f : M_n(B(H)) \to B(H^{(n)})$ which is the $*$-isomorphism? It is the function that takes a matrix $a = [T_{ij}]$ in $M_n(B(H))$, that is a matrix whose entries are operators T_{ij}, to the operator L_a from $H^{(n)}$ to $H^{(n)}$ described as follows.

$$L_a\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = \begin{bmatrix} T_{11}(x_1) + T_{12}(x_2) + \cdots T_{1n}(x_n) \\ T_{21}(x_1) + T_{22}(x_2) + \cdots T_{2n}(x_n) \\ \vdots \\ T_{n1}(x_1) + T_{n2}(x_2) + \cdots T_{nn}(x_n) \end{bmatrix} , \qquad \forall \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in H^{(n)} \,.$$

How should you understand the right hand side of the last equation: it is just the 'matrix product' of the matrix $[T_{ij}]$ and the column vector $[x_j]$. That is, it is just the 'matrix product'

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} .$$

Now we understand what the function $f(a) = L_a$ is, and we also see that it is a generalization of the f we used in the proof of Equation (1). The proof that f is a $*$-isomorphism is almost identical to the proof we gave in the case of Equation (1).

Exercise: Prove it!

Chapter 2

A little about Banach spaces and C^*-algebras

2.1 Reminder on normed spaces

Reminder: A *norm* on a vector space V is a function $\|\cdot\| : V \to [0,\infty)$ satisfying the following properties:

(i) $\|x\| \geq 0$ for all $x \in V$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{F}$ and $x \in V$,

(iii) (Triangle inequality) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$,

(iv) $\|x\| = 0$ implies that $x = 0$.

If $\|\cdot\|$ is a norm on a vector space V, then we say that $(V, \|\cdot\|)$ is a *normed vector space* (or normed linear space, or *normed space*).

We wrote $B(X,Y)$ for the set of bounded (i.e. continuous) linear operators from X to Y, when X and Y are normed spaces. Then $B(X,Y)$ is also a normed space with the norm $\|T\|$ which we defined. A special case of particular interest is when Y is just the scalars; we write X^* for $B(X,\mathbb{R})$, and call this space the *dual space* of X. The functions in X^* are called *functionals*.

Thus if X is a normed space then X^* is a normed space, with norm $\|f\| = \sup\{|f(x)| : x \in \text{Ball}(X)\}$. Because X^* is a normed space, we can look at its dual space too. We write $(X^*)^*$ as X^{**}. It too is a normed space. It is very important that there is a canonical function from X into $(X^*)^* = X^{**}$. We will write this function as i_X or as $\hat{\ }$. We have

$$i_X(x)(f) \;=\; \hat{x}(f) \;=\; f(x)\;, \qquad x \in X, f \in X^*.$$

It is easy to see that $i_X(x)$ is indeed in $(X^*)^*$: for example,

$$i_X(x)(f+g) \;=\; (f+g)(x) \;=\; f(x)+g(x) = i_X(x)(f) + i_X(x)(g).$$

and

$$|i_X(x)(f)| \ = \ |f(x)| \ \leq \ \|f\|\|x\|,$$

so that $i_X(x)$ is bounded. Indeed the last line shows that $\|i_X(x)\|$, which we recall is defined to be $\sup\{|i_X(x)(f)| : f \in \text{Ball}(X^*)\}$, is just:

$$\|i_X(x)\| = \sup\{|f(x)| : f \in \text{Ball}(X^*)\} \ \leq \ \|x\| \ , \qquad \forall x \in X.$$

In other words, this function i_X from X into X^{**} is a contraction. In fact i_X is an *isometry*. That is, $\|i_X(x)\| = \|x\|$, $\forall x \in X$. This one is not so easy. We will show that it follows from another result, one of the most important theorems in the subject of functional analysis. This is the *Hahn-Banach theorem.*

2.2 The Hahn-Banach theorem

The following is perhaps the best known version of this theorem:

Theorem 2.2.1 (The Hahn-Banach theorem) *Given any linear subspace Y of a Banach space X, and any bounded linear functional $f \in Y^*$, there exists a bounded linear $\tilde{f} \in X^*$ extending f (that is, such that $\tilde{f}(y) = f(y)$ if $y \in Y$). Indeed this may be done with $\|f\| = \|\tilde{f}\|$.*

I will not prove this result here. Although the proof is not difficult, it is long, and would take too much of our time together. Instead, we will talk about consequences, and later we will talk about the 'noncommutative generalization' of this theorem.

- As a first consequence of the Hahn-Banach theorem, I will prove that the function i_X we discussed earlier, is an isometry (that is, $\|i_X(x)\| = \|x\|$, $\forall x \in X$). This means that we can think of X as a subspace of its second dual X^{**}, which is very important!

  So take any $x \in X$. We may assume $x \neq 0$, otherwise the result is obvious. The set Y of all scalar multiples of x is a subspace of X, and so it is a normed space. In fact $\|cx\| = |c|\|x\|$, for any scalar c. Define a function g on Y by $g(cx) = c\|x\|$. It is easy to see that this scalar valued function g is linear (Exercise: check it!), and by one of the formulae we gave earlier for the norm of a linear function,

  $$\|g\| = \sup\{\frac{|g(cx)|}{\|cx\|} : cx \neq 0\} = \sup\{\frac{|c\|x\||}{|c|\|x\|} : c \neq 0\} = 1.$$

  By the Hahn-Banach theorem, there is a bounded linear $\varphi \in X^*$ with $\|\varphi\| = \|g\| = 1$, and such that $\varphi(x) = g(x) = \|x\|$. Thus $i_X(x)(\varphi) = \varphi(x) = \|x\|$, and hence

  $$\|i_X(x)\| \ = \ \sup\{|f(x)| : f \in \text{Ball}(X^*)\} \ \geq \ |\varphi(x)| \ = \ \|x\|.$$

  Since we saw earlier that $\|i_X(x)\| \leq \|x\|$, we have proved that $\|i_X(x)\| = \|x\|$.

- As a second consequence of the Hahn-Banach theorem, I am going to show that there are some normed spaces which are most important! I will also show why normed spaces can be considered to be 'commutative' objects. To see all this we will need to introduce two new notations:

$$C(K) - \text{spaces}$$

and

$$\ell^\infty(S)$$

First let me discuss $\ell^\infty(S)$. We take a set S, and let S be all the bounded functions from S to the scalar field. Recall that a function is bounded if there is a number M such that $|f(x)| \leq M$, $\forall x$. In a previous lecture we defined

$$\|f\|_\infty \;=\; \sup\{|f(x)| : x \in S\}.$$

Thus

$$\ell^\infty(S) \;=\; \{f : S \to \text{scalars} : \|f\|_\infty < \infty\}.$$

It is easy to check that $\ell^\infty(S)$ is a normed space, with the norm $\|f\|_\infty$. (Exercise: show it.) For example, the most difficult part is to check the 'triangle inequality' $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. But

$$\begin{aligned} \|f+g\|_\infty &= \sup\{|f(x)+g(x)| : x \in S\} \\ &\leq \sup\{|f(x)| + |g(x)| : x \in S\} \\ &\leq \sup\{|f(x)| : x \in S\} + \sup\{|g(x)| : x \in S\} \\ &= \|f\|_\infty + \|g\|_\infty. \end{aligned} \tag{2.2.2.1}$$

Because $\ell^\infty(S)$ is a normed space, so is every subspace of $\ell^\infty(S)$. Thus we can get a huge list of normed spaces by looking at all subspaces of $\ell^\infty(S)$. I claim that in fact EVERY normed space is on this list!!! This makes $\ell^\infty(S)$ is a 'most important normed space'.

Why is EVERY normed space a linear subspace of $\ell^\infty(S)$? To prove this, let X be any normed space, and let S be the set $\text{Ball}(X^*)$. Define a function $j : X \to \ell^\infty(S)$ by

$$j(x)(f) \;=\; f(x)\,, \qquad \forall\, f \in \text{Ball}(X^*), x \in X.$$

Note that

$$\|j(x)\|_\infty \;=\; \sup\{|j(x)(f)| : f \in S\} \;=\; \sup\{|f(x)| : f \in \text{Ball}(X^*)\} \;=\; \|i_X(x)\| \;=\; \|x\|.$$

Thus we see, first, that $j(x) \in \ell^\infty(S)$ as desired, and also, j is an isometry. Thus X is isometric to the range of j. That is, we may identify X and the range of j, which is a subspace of $\ell^\infty(S)$.

- Lets now talk about $C(K)$-spaces, and then do a small variation of the last argument. Let K be a compact topological space. If you are unfamiliar with the notion of topological spaces, just think of it as a set K together with a collection of subsets of K which we have decided to call 'open'; and this collection must have 3 or 4 properties reminiscent of the properties of open sets in $\mathbb{R}^2$ for example, e.g. the union of open sets is open. Once you have a topology it makes sense to talk about compactness, continuity, etc. If you like, just take K below to be a compact (i.e. closed and bounded) subset of $\mathbb{R}^n$. This is only one special class of compact spaces, but it gives a good picture. For any compact topological space K, we may consider the set $C(K)$ of continuous scalar valued functions f on K. Again define $\|f\|_\infty = \sup\{|f(x)| : x \in K\}$. Its easy to check that $C(K)$ is a normed space with this norm $\|f\|_\infty$.

- The space $C(K)$ has a lot of algebraic structure. Firstly, it is a vector space, because the sum of two continuous functions is continuous. Then it has a product fg of elements $f, g \in C(K)$ (namely $(fg)(x) = f(x)g(x)$ for $x \in K$). Thus $C(K)$ is a commutative algebra. Indeed it is a $*$-algebra, if we define $f^*(x) = \overline{f(x)}$ for $x \in K$.

  Similar things are true about $\ell^\infty(S)$.

- Subspaces of $C(K)$ are normed spaces, and so again we get a huge list of normed spaces if we look at linear subspaces of $C(K)$-spaces. In fact this list again includes every normed space!! To see this, we will need to introduce a 'topology' on X^*, and therefore on the subset Ball(X^*), called the 'weak* topology'. The weak* topology on X^* is defined to be the smallest topology on X^* for which all the functions $i_X(x)$ above are continuous. You will not need to know anything about the weak* topology except a) all the functions $i_X(x)$ above are continuous, and b) Ball(X^*) is compact in this topology (this is a theorem due to Alaoglu, which we will not have time to prove. Just take it on faith).

  Given any normed space X, let $K = \text{Ball}(X^*)$ with its weak* topology. By (b) above, K is compact. The isometric function $j : X \to \ell^\infty(K)$ above, actually goes into $C(K)$, by (a) above. Thus X is isometric to the range of j. That is, we may identify X and the range of j, which is a subspace of $C(K)$.

- Summary: Both $\ell^\infty(S)$ and $C(K)$ are commutative $*$-algebras. Also, every normed space may be viewed as a subspace of these. Thus every normed space may be viewed as consisting of commuting functions on S or K.

2.3 An algebraic formulation of topology: more on $C(K)$-spaces

If you havent met the notion of a general topological space, thats OK. Just think of a topological space as a set K together with a collection of subsets of K which we have decided to call 'open'; and this collection must have 3 or 4 properties reminiscent of the properties of open sets in $\mathbb{R}^2$ for example, e.g. the union of open sets is open. Once you have a topology it makes sense to talk about compactness, continuity, etc. We will just talk here about compact topological spaces, and if you like just think about compact sets in $\mathbb{R}^n$.

We saw above that $C(K)$ is a $*$-algebra, and also is a normed space. An important feature of $C(K)$-spaces, is that the space K is essentially recoverable from the algebra $C(K)$. This is how to do it: for any commutative algebra A define a character of A to be a homomorphism $\chi : A \to \mathbb{C}$ with $\chi(1) = 1$. We write $A^\#$ for the set of characters of A. If $A = C(K)$, and if $x \in K$, then the function $f \mapsto f(x)$ is a character of A. Call this character χ_x. It is striking that the converse is true: every character of A equals χ_x for some unique point $x \in K$. This we will not prove. In any case, we see that $A^\#$ is in a one-to-one correspondence with K. In addition, clearly $A^\# \subset \text{Ball}(X^*)$, since if x is the point associated with a character χ as above,

$$|\chi(f)| = |f(x)| \leq \sup\{|f(w)| : w \in K\} = \|f\|_\infty.$$

Thus $A^\#$ gets a topology, namely the weak* topology from X^*. The remarkable thing is that the function $x \mapsto \chi_w$ above, from K to $A^\#$, is a 'homeomorphism' (it is one-to-one, onto, continuous, and its inverse is continuous. Thus as topological spaces, $A^\#$ 'equals' K. Thus we may retrieve the topological space K (up to homeomorphism) from the algebra $C(K)$, namely, $K = A^\#$. Thus we have a way of going from a compact space K, to an algebra $C(K)$, and a way of going from the algebra $C(K)$ back to the space K, and these two 'operations' are inverses to each other: $K \cong C(K)^\#$ as topological spaces, and $C(K) \cong C(C(K)^\#)$ as algebras.

Actually, the correspondence in the last paragraph is just the tip of a beautiful iceberg. It shows that the study of compact spaces K, is the same as the study of the commutative $*$-algebras $C(K)$. Thus, every topological property in a compact space K, must be reflected by an algebraic property in the algebra $C(K)$. For example, let me prove to you that K is connected iff the algebra $C(K)$ contains no nontrivial idempotents (i.e. no p except 0 and 1, such that $p^2 = p$). To see this, suppose that $p \in C(K)$ with $p^2 = p$. Then for any $x \in K$, $p(x)^2 = p(x)$. The only scalars z such that $z^2 = z$ are 0 or 1, so therefore $p(x)$ equals 0 or 1. Let $U = \{x \in K : p(x) < \frac{1}{2}\}$ and $V = \{x \in K : p(x) > \frac{1}{2}\}$. These are open since p is continuous, disjoint, and not empty if p is not always 1 or always 0. Thus K is disconnected. The other direction of the 'iff' is easier, and follows by reversing the argument.

We can make things even prettier, by removing all mention of K. This was accomplished by the mathematician Gelfand. He noticed that $C(K)$ has a peculiar property. It is a $*$-algebra with a norm satisfying the following two conditions for any $f, g \in C(K)$:

$$\|fg\| \leq \|f\|\|g\|$$

and

$$\|f\bar{f}\| = \|f\|^2.$$

The latter is called the *C^*-identity*. Lets prove it: for the first we note that for any $f, g \in C(K)$:

$$\|fg\| = \sup\{|f(x)g(x)| : x \in K\} \leq \sup\{|f(x)| : x \in K\} \sup\{|g(x)| : x \in K\} = \|f\|\|g\|.$$

For the second notice that since $z\bar{z} = |z|^2$ for scalars z, we have

$$\|f\bar{f}\| = \sup\{|f(x)\overline{f(x)}| : x \in K\} = \sup\{|f(x)|^2 : x \in K\} = \sup\{|f(x)| : x \in K\}^2 = \|f\|^2.$$

We will call a $*$-algebra with these properties a C^*-algebra. Thus $C(K)$ is commutative C^*-algebra. Remarkably, the converse is true, and this is called Gelfand's theorem, any commutative C^*-algebra A (with a 1) is isometrically *-isomorphic (i.e. isomorphic in every possible way) to a $C(K)$, for some compact space K.

Thus these commutative C^*-algebras are exactly the $C(K)$-spaces. Putting this together with our earlier comments, we see that studying (compact, say) topological spaces K, is the same as studying these commutative C^*-algebras.

2.4 A few facts about general C^*-algebras

A C^*-algebra is a $*$-algebra A, with a complete norm satisfying $\|xy\| \leq \|x\|\|y\|$, and also the so-called *C^*-identity*: $\|x^*x\| = \|x\|^2$, for all $x, y \in A$. Think of them of being comprised of noncommutative numbers. In fact the norm is given by the formula:

$$\|a\| = \sqrt{\|\text{the largest spectral value of} a^*a},$$

where the spectral values of b are the numbers λ such that $\lambda 1 - b$ is not invertible.

The most important functions between C^*-algebras are the $*$-homomorphisms. A remarkable fact is that:

Theorem 2.4.1 *Any $*$-homomorphism between C^*-algebras is automatically contractive, and any one-to-one $*$-homomorphism between C^*-algebras is automatically isometric.*

Proof: I'll just prove the last statement, the first one being similar. So suppose that $\theta : A \to B$ is a $*$-homomorphism between two C^*-algebras which is one-to-one and onto. This means that algebraically, A and B are the same. So a is positive in A if and only if $\theta(a)$ is positive in B, and if λ is a scalar then $\lambda 1 - a$ is invertible if and only if $\lambda 1 - \theta(a) = \theta(\lambda 1 - a)$ is invertible. So the spectral values of a and $\theta(a)$ are the same. Since the norm is defined to be the largest spectral value, $\|a\| = \|\theta(a)\|$. If a is not positive, then using the C^*-identity, and the last line applied to a^*a (which is positive), we have

$$\|a\| = \sqrt{\|a^*a\|} = \sqrt{\|\theta(a^*a)\|} = \sqrt{\|\theta(a)^*\theta(a))\|} = \|\theta(a)\|,$$

which says θ is isometric. $\square$

Thus any $*$-isomorphism between C^*-algebras is automatically an isometric $*$-isomorphism. Thus we think of two C^*-algebras as being 'the same' if they are $*$-isomorphic. We have the important principle that:

There is at most one 'good' norm on a *-algebra.

Here 'good' means a C^*-algebra norm. In fact this norm is given by the formula:

$$\|a\| \stackrel{def}{=} \sqrt{\|\text{the largest spectral value of} a^*a.}$$

Thus C^*-algebras are quite 'rigid' objects. We have already seen many examples of C^*-algebras in this course. The scalar field is itself a C^*-algebra, with $\|z\| = |z|$. We saw that $C(K)$ spaces are commutative C^*-algebras, and vice versa. Also, $\ell^\infty(S)$ is a commutative C^*-algebra (by the argument we used to show that $C(K)$ is a C^*-algebra). One can verify that the $*$-algebra M_n of $n \times n$ matrices is a C^*-algebra, for any $n \in \mathbb{N}$. Also $B(H)$ is a C^*-algebra for any Hilbert space H. Let us check this: we know that $B(H)$ satisfies $\|ST\| \leq \|S\|\|T\|$ (indeed we proved this earlier). Lets check the C^*-identity: First, note that if $T \in B(H)$ and $x \in \text{Ball}(H)$ then

$$\|Tx\|^2 = \|\langle Tx, Tx\rangle\| = \|\langle x, T^*Tx\rangle\| \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|\|x\| \leq \|T^*T\| \leq \|T^*\|\|T\|,$$

the first '$\leq$' by the Cauchy-Schwarz inequality. Thus

$$\|T\|^2 = \sup\{\|Tx\|^2 : x \in \text{Ball}(H)\} \leq \|T^*T\| \leq \|T^*\|\|T\|.$$

Dividing by $\|T\|$, we see that $\|T\| \leq \|T^*\|$. Replacing T by T^* we see that $\|T^*\| \leq \|T\|$, since $(T^*)^* = T$. Hence

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| \leq \|T\|^2,$$

which gives the C^*-identity. Thus $B(H)$ is a C^*-algebra.

¿From this it follows immediately that every (closed) $*$-subalgebra of $B(H)$, where H is a Hilbert space, is a C^*-algebra. (A *subalgebra* of an algebra is just a linear subspace $B \subset A$ such that $ab \in B \; \forall a, b \in B$; it is again an algebra. A $*$-*subalgebra* of a $*$-algebra is a subalgebra B such that $b^* \in B \; \forall b \in B$; it is again a $*$-algebra.) A famous theorem due to Gelfand and Naimark says that the converse is also true:

Theorem 2.4.2 (Gelfand-Naimark) *Every C^*-algebra is ($*$-isomorphic to) a norm-closed $*$-subalgebra of $B(H)$, for some Hilbert space H.*

This is the 'noncommutative version' of Gelfand's theorem which we mentioned earlier.

One can show that $B(H)$, for a Hilbert space H, has a predual Banach space. That is, there is a Banach space Y such that $Y^* \cong B(H)$ isometrically. We will say a little more about this in the next chapter. A *von Neumann algebra* M is a $*$-subalgebra of $B(H)$ which is closed in the weak* topology of $B(H)$. A well-known theorem due to Sakai says that von Neumann algebras may be characterized as the C^*-algebras which have a predual Banach space. The commutative von Neumann algebras may be abstractly characterized as the L^∞ spaces (if you know what L^∞ means).

Application: We said above that if a $*$-algebra A is $*$-isomorphic to a C^*-algebra, then there can be only one norm on A making A a C^*-algebra. Now $M_n(B(H))$ is a $*$-algebra (see the end of Chapter 1). The 'good' C^*-algebra norm on $M_n(B(H))$ is the one that forces the $*$-isomorphism

$$M_n(B(H)) \cong B(H^{(n)})$$

which we studied at the end of Chapter 1, to be an isometry. That is, if $a \in M_n(B(H))$, then

$$\|a\| = \sup\{\|L_a(x)\| : x \in \mathrm{Ball}(H^{(n)})\},$$

where L_a is as we defined it in that earlier lecture.

We will always consider this norm on $M_n(B(H))$.

2.5 Applications to norms of matrices

¿From the C^*-identity, we can quickly deduce several important properties of norms of matrices.

- First, note that the norm of a diagonal matrix $D = diag\{d_1, d_2, \cdots, d_n\}$ is easy to find: it is the square root of the biggest eigenvalue of $D^*D = diag\{|d_1|^2, |d_2|^2, \cdots, |d_n|^2\}$. The eigenvalues are the numbers on the diagonal, so we see that the biggest eigenvalue of D^*D is $\sup\{|d_k|^2\}$. So the norm of D is $\sup\{|d_k|\}$.

  Indeed the space of all diagonal matrices D is a commutative C^*-algebra, isometrically $*$-isomorphic to $\ell^\infty(S)$, where S is an n point set.

- Next, lets compute the norm of the matrix

$$C = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_n & 0 & \cdots & 0 \end{bmatrix}.$$

  By the C^*-identity, we know that $\|C\|^2 = \|C^*C\|$. But

$$C^*C = \begin{bmatrix} \overline{a_1} & \overline{a_2} & \cdots & \overline{a_n} \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_n & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \sum_k |a_k|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

  which is diagonal, and has norm $\sum_k |a_k|^2$. Thus $\|C\| = \sqrt{\sum_k |a_k|^2}$. This shows that the space of all such matrices C (which are all zero except in the first column), is the n-dimensional Euclidean space (with its 2-norm).

- Similar calculation shows that

$$\left\| \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\| = \sqrt{\sum_k |a_k|^2}.$$

This shows that the space of all matrices which are all zero except in the first row, is the n-dimensional Euclidean space (with its 2-norm).

- Let U be a unitary. By the C^*-identity,

$$\|Ux\| = \sqrt{\|(Ux)^*Ux\|} = \sqrt{\|x^*U^*Ux\|} = \sqrt{\|x^*Ix\|} = \|x\|.$$

Similarly, $\|xU\| = \|x\|$.

Let us apply the last principle, to show that switching around rows or columns in a matrix, does not change its norm. For example, the matrix

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

And the matrix of 0's and 1's here is a unitary. Thus

$$\left\| \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right\|.$$

- Adding extra rows (or columns) of zeros to a matrix, does not change its norm either.

To see this, lets suppose that A is a matrix, and B is A with several extra rows of zeros added. We claim that $\|A\| = \|B\|$. By switching around rows, we can assume that all the extra rows of zeros added, are at the bottom of B. Then $B^*B = A^*A$, where 0 here is a matrix of zeros (check this, by writing out a simple example). Thus by the C^*-identity,

$$\|B\|^2 = \|B^*B\| = \|A^*A\| = \|A\|.$$

Thus when finding norms, one may always if one likes, assume that the matrix is square (by adding extra rows (or columns) of zeros).

- The 'direct sum' of matrices. If A and B are square matrices (of possibly different sizes, we define $A \oplus B$ to be the matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Here the 0's are actually matrices of zeros. In fact

$$\|A \oplus B\| = \max\{\|A\|, \|B\|\}.$$

Exercise: Prove this.

- One final result about norms of matrices: Consider the function $g : M_n \to M_n(B(H))$ taking a matrix $a = [a_{ij}]$ of scalars to the matrix $[a_{ij}I]$ of operators. Here I is the 'identity function on H', that is $I(x) = x$ for all $x \in H$. It is easy to check that g is an isometry (Exercise—one way to do it is to show that g is a one-to-one $*$-homomorphism, and then use Theorem 2.4.1). If $x = [T_{ij}] \in M_n(B(H))$, that is, if $[T_{ij}]$ is a matrix whose entries are operators, then we define ax to be the product $g(a)x$ in the algebra $M_n(B(H))$. Similarly, define $xa = xg(a)$. Then since $M_n(B(H))$ is a C^*-algebra,

$$\|ax\| = \|g(a)x\| \leq \|g(a)\|\|x\| = \|a\|\|x\|.$$

  Similarly, $\|xa\| \leq \|x\|\|a\|$.

Chapter 3

Noncommutative mathematics

3.1 The noncommutative world

¿From the beginning of the 20th century, new and noncommutative mathematical phenomena began to emerge, in large part because of the emerging theory of quantization in physics. Heisenberg phrased his quantum physics in terms of infinite matrices. Such matrices replace the time dependent variables. So from the beginning there was an emphasis on matrices. The scalar valued functions of Newtonian physics, are replaced by matrices; and generally one should think of a matrix as the 'quantized' or 'noncommutative version' of a scalar valued function. The 'values' of a matrix are given by its eigenvalues, and its spectrum (i.e. set of eigenvalues). Note that these are defined by an *algebraic* statement about whether $\lambda I - A$ has an inverse in a certain algebra.

The work of many mathematicians and mathematical physicists (such as John von Neumann) on quantization, suggested that the observables in quantum mechanics be regarded as self-adjoint matrices, or indeed self-adjoint operators on a Hilbert space (recall that $M_n \cong B(H)$ for a Hilbert space H). He and Murray, in the 30's and 40's, introduced what are now known as von Neumann algebras, which are a very important class of *-subalgebras of $B(H)$, and which also are the noncommutative' version of the theory of integration and the integral (we'll discuss this shortly). Gelfand's work showed that C^*-algebras are the 'noncommutative topological spaces' as we saw last lecture. And C^*-algebras became important (to some) in quantum physics and quantum field theory.

Thus we have the classical 'commutative world', of functions and function spaces, and also the 'noncommutative world' of matrices, operators on Hilbert space, and C^*-algebras and other important *-algebras. Loosely speaking, and this is no doubt a not quite proper usage, we use the word 'quantized' for this noncommutative world. So a matrix is a 'quantized function', a C^*-algebra is a 'quantized topological space', and so on.

It is important to bear in mind that correct 'noncommutative mathematics' should always be a GENERALIZATION of the classical 'commutative case'. For example, if you make an assertion such as 'Property P is the noncommutative or quantized version of the classical Property Q'. Then you must be sure that if you take the classical object and view it as an object in the noncommutative world, then it has Property P if and only if the original object had Property Q.

3.2 The basic strategy of 'noncommutative mathematics'

The following is a basic 6-step strategy commonly encountered in 'noncommutative mathematics' (we will go into more detail on these steps in a moment, in specific examples): Namely, the first step is to point out that studying several of the commonly encountered spaces in mathematics (e.g. the spaces one meets in topology, measure and integration theory, probability, differentiation, manifolds, groups), is the same as studying algebras of appropriate functions on these spaces (e.g. $C(K)$, certain algebras of measurable functions, $C^\infty(K)$, etc.). The second step is to replace these commutative algebras by noncommutative ones having the same formal properties. The third step is to find lots of good examples of such noncommutative algebras, which do arise and are important in physics and mathematics. Fourth, one generalizes the analysis which arises in the commutative case, to the noncommutative. Fifth, one usually needs to also develop the noncommutative analysis in other ways too, besides what you see in the 'classical'/commutative case. In practice, it is startlingly beautiful to see how these work out! There really is a 'noncommutative world out there'! Thus there are now important, deep, and beautiful theories of 'noncommutative topology', 'noncommutative probability', 'noncommutative differential geometry', 'quantum groups', 'noncommutative functional analysis', and so on. The sixth step is to use these theories to solve important problems.

We have already seen the beginnings of 'noncommutative topology'. We saw earlier that there is a perfect correspondence between the topology of a space K (i.e. the open and closed sets, compactness, connectedness, etc), and the algebraic structure of the algebra $C(K)$. This is the first step in our strategy above. Second, one may summarize the properties of $C(K)$-spaces, by a list of axioms—namely those in the definition of a 'commutative C^*-algebra', and then we remove the 'commutativity' assumption, that is we see that we have to study general C^*-algebras. Third, one then looks for examples of noncommutative C^*-algebras that are important elesewhere in math and physics, such as $B(H)$. Fourth, we generalize many important things one sees in topology, and which are reflected algebraically in the $C(K)$ algebras, to general C^*-algebras. For example, studying closed subsets of $[0, 1]$ say, corresponds to studying quotients of the associated algebra $C([0, 1])$ by a closed 'ideal'. Fifth, one develops the general theory of C^*-algebras in other ways. Some of these ways do not show up in the commutative world, but are nonetheless important in math or physics. Sixth, one solves problems!

In fact the idea of replacing a geometric space by an algebra of functions on it, and working instead with that algebra is an old one. It is a common perspective in algebraic geometry for example. Also, one of the main theorems of the last century is the Atiyah—Singer index theorem (which you will have another lecture series on soon), and all the associated theory of pseudodifferential operators and manifolds, becomes naturally connected to certain *-algebras. A fundamental tool in topology, K-theory, is defined in terms of vector bundles; however it can be equivalently formulated in terms of modules over $C(K)$, and the theory and theorems are often best proved by using algebra methods. All of this was very suggestive in leading to much more sophisticated noncommutative topology and noncommutative geometry. Thus Connes says "... K-theory and K-homology admit noncommutative geometry as their natural framework".

3.3 Noncommutative integration theory

When one looks at von Neumann algebras it is very clear that they are a far reaching noncommutative generalization of classical integration theory. Classical integration theory, for example of the Lebesgue integral, begins with a set $\mathcal{B}$ of subsets of a set K, called *measurable sets*, and a *measure*, that is a function $\mu : \mathcal{B} \to [0, \infty)$ which assigns a 'measure', or 'size', or 'volume', to each set. However very quickly one starts to work with the characteristic functions 1_E of the sets, instead of with the sets themselves. Indeed one works with the linear combinations $\sum_k c_k 1_{E_k}$ of such characteristic functions, which are called *simple functions*. The set of simple functions is clearly a commutative $*$-algebra. And instead of working with the measure, one works much more with the *integral* $\int f$. If you have studied more advanced integration theory, you may know that there is a theorem which says that you dont really need the measure at all, you can if you like just work with the integral. That is, instead of beginning with a measure, begin with a linear functional φ on $C(K)$ say, and define $\int f = \varphi(f)$. ¿From there you can build the integral of noncontinuous functions, and get everything one needs. The set of functions whose integral is finite, the *integrable functions* is called L^1. It is a normed space whose dual space is L^∞, the functions which are bounded except on a negligable set. It turns out that L^∞ is a commutative C^*-algebra. Thus we have the first step of the 'general strategy' listed above, we have replaced the classical measure and integration theory, by something 'equivalent, the commutative C^*-algebra L^∞. The second step in the strategy is to ask what is the key property that the commutative C^*-algebra L^∞ has? In this case it is a commutative C^*-algebra with a predual (namely L^1). Conversely, one can prove that every commutative C^*-algebra with a predual 'is' an L^∞. So the key property is that it is a commutative C^*-algebra with a predual. Thus in the second step in the strategy, we replace commutative C^*-algebras with a predual, by general C^*-algebras with a predual. As we said in the last chapter, these are exactly the von Neumann algebras. Thus von Neumann algebras should be regarded as the 'noncommutative L^∞-spaces', and their theory is therefore 'noncommutative integration theory'. The integral has been replaced by a functional on the von Neumann algebra (or something akin to it). For example, M_n is a von Neumann algebra, its predual is M_n but with a different norm $\|\cdot\|_1$, called the *trace norm* because if A is a positive matrix then $\|A\|_1$ is just the trace of A. Indeed the trace $tr(A) = \sum_k a_{kk}$ takes the place of the integral in this example. The third step in the strategy, is then to find important von Neumann algebras in math and physics. One does'nt have to look far, indeed the reason von Neumann developed von Neumann algebras was to give a mathematical foundation for quantum physics!! The fourth step in the strategy, is to generalize the classical integration theory. Their theory turns out to be quite intricate and deep, rather beautiful, and extremely powerful. There are some surprises: some genuinely 'noncommutative phenomena' appear which were not visible in the classical 'commutative' integration theory (this is the fifth step).

One may argue against this ideology as follows: "OK, I agree that a von Neumann algebra appears to be a good analogue of an L^∞-space, and what happens in the commutative case is rather compelling. But the definition of a von Neumann algebra seems rather complicated and restrictive. Maybe other noncommutative *-algebras could also be good 'noncommutative L^∞-spaces'. But in fact there does not appear to be any better class of $*$-algebras to use to define noncommutative measure theory. It is clear from the classical 'commutative' integration theory, you need lots of projections around to do integration theory. One can prove using the famous 'Spectral Theorem' that von Neumann algebras have lots of projections, in fact it is the closed span of its projections. Also you need duality and the weak* topology to do much in measure theory (although in classical integration theory the weak* topology is given a different name). For these reasons it seems fairly clear that von Neumann algebras are the best candidates for 'noncommutative L^∞-spaces'.

3.4 Noncommutative probability theory

It is a well known principle that probability theory can be described as 'measure theory (i.e. integration theory) plus the concept of independence'. Therefore noncommutative probability theory should be the study of von Neumann algebras and an accompanying 'noncommutative independence'. There is a large and quite recent such theory, in large part due to D. Voiculescu. A major tool in this theory is 'random matrices' and the distribution of their eigenvalues. But one also needs a lot of von Neumann algebra theory, for the reasons outlined above.

3.5 Quantum groups

The field called 'harmonic analysis' or 'Fourier analysis' is a large area of mathematics. The usual framework for studying this subject, is a group G which has a topology, so that the group operations (i.e. the multiplication and the 'inverse' $g \mapsto g^{-1}$) are continuous. For example the unit circle $\mathbb{T}$ in the complex plane, with its usual 'arclength metric', is a compact group. It is important that one can prove that there is a very special and unique measure, called Haar measure, around; for example, in $\mathbb{T}$ it is the length of an arc of the circle. Using this Haar measure one gets the Fourier transform, and the Fourier analysis.

Let us now look at the 'noncommutative version' of a compact group. First, observe that studying $A = C(G)$ as an algebra is not enough to capture the group operations. To encode the product $(g, h) \mapsto gh$, which is a function from $G \times G \to G$, we replace it by the function $\Phi : C(G) \to C(G \times G)$ which takes a function $f \in C(G)$ to the function

$f(gh)$ of two variables $(g, h) \in G \times G$. This function is called the 'comultiplication'. The associativity of the group product is captured by a certain commutative diagram for Φ. In fact $C(G \times G) = C(G) \otimes C(G)$ (the latter taken to mean the 'completion of the algebraic tensor product in a certain tensor norm'). Thus a 'noncommutative compact group' should be a unital C^*-algebra A, together with a linear function $\Phi : A \to A \otimes A$ satisfying a certain commutative diagram.

Again, this at first sight looks like a fairly ad hoc definition. But from it one can prove the existence of a Haar measure, Fourier transform, etc; in other words one has a noncommutative theory which is a startling and far reaching generalization of the usual theory of compact groups. The usual theorems and development can be redone now in a noncommutative and far more general setting. And examples of such noncommutative C^*-algebras appear in physics.

3.6 Noncommutative geometry, etc

Then there is 'noncommutative differential geometry', mostly due to Connes. One idea here is that studying a differential manifold M should be equivalent to studying the algebra $C^\infty(M)$ of infinitely smooth functions. This is not a *-algebra, but it is dense in a C^*-algebra. So a 'noncommutative manifold' should be a certain class of C^*-algebras which possess a special kind of dense subalgebra of 'smooth elements'. However now this theory of noncommutative differential geometry is much more advanced and includes a 'quantized calculus'. See A. Connes incredibly deep book "Noncommutative geometry".

There is also a theory of 'noncommutative metric spaces', 'noncommutative dynamical systems', etc.

3.7 Noncommutative normed spaces

Finally we turn to my main field of interest, operator spaces, which may be described as 'noncommutative normed spaces' or 'noncommutative functional analysis'. Sometimes it is called 'quantized functional analysis'. We want to apply our 'six step quantization strategy' to normed spaces and their theory. In order to thoroughly explain the first two steps, we will need to explain a few important definitions and results.

- A (concrete) *operator space* is just a linear subspace X of $B(H)$, for a Hilbert space H. Remembering that $B(H)$ is just a space of matrices M_n (with n possibly infinite), we see that (if you wanted to-but we usually dont) an operator space may be regarded as a vector space whose elements are matrices. Every operator space X is a normed space, because $B(H)$ or M_n have a norm which X inherits. However an operator space has more than just a norm. To see this, remember the important principle:

  A matrix of operators is an operator!

Thus if X is an operator space, then an $n \times n$ matrix $[x_{ij}]$ whose entries are elements of X, can be regarded as an operator, and so it too has a norm. We write this norm as $\|[x_{ij}]\|_n$.

Here is another way to say it: if $X \subset B(H)$ then $M_n(X) \subset M_n(B(H))$, obviously. But we saw that $M_n(B(H))$ had a (unique) C^*-algebra norm. Therefore $M_n(X)$ gets this natural norm.

$$X \subset B(H) \quad \Rightarrow \quad M_n(X) \ \subset \ M_n(B(H)) \cong B(H^{(n)}) \ .$$

The matrix norms are very important. In fact: *The norm alone, on the operator space, often does not contain enough information to be helpful in 'noncommutative functional analysis'.*

- To illustrate, lets look at two very important operator spaces which live inside M_n, called C_n and R_n. We have already met them:

$$C_n \;=\; \begin{bmatrix} * & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ * & 0 & \cdots & 0 \end{bmatrix} \qquad ; \qquad R_n \;=\; \begin{bmatrix} * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix} .$$

ADD better to do as commuting triangle, and take matrix in $M_2(\mathbb{R}^2)$. Consider the following matrix in $M_2(C_2)$:

$$A \;=\; \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} .$$

This matrix A has norm 1 in $M_2(C_2)$ because removing 'inner matrix brackets', and removing 'rows and columns of zeros, gives

$$\|A\|_2 \;=\; \left\| \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \right\| \;=\; \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\| \;=\; \|I_2\| \;=\; 1.$$

Now A is not in $M_2(R_2)$. However remember from page 26 that as normed spaces, C_2 and R_2 are 'the same', they are both equal to the 2 dimensional Euclidean space. Indeed we noticed on page 26 that the formula for the norm of a matrix in C_n, and the formula for the norm of a matrix in R_n, are the same. This is saying that the 'identity' function from C_2 to R_2 (the transpose), is an isometry. The matrix in $M_2(R_2)$ corresponding to A via this function, is

$$B = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

This matrix B has norm $\sqrt{2}$ because removing 'inner matrix brackets', and removing 'rows and columns of zeros, gives (see p. 26),

$$\|B\|_2 = \left\| \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\| = \sqrt{2}\,.$$

- **The main point:** If X is an operator space, then $M_n(X)$ has a natural norm too, for each $n \in \mathbb{N}$. (We just calculated these norms in $M_2(X)$ in particular cases). In operator space theory we make the commitment to keep track of (or at least be aware of) these 'matrix norms' too. Because we no longer just have one norm to deal with, but also the matrix norms, the following definitions (due to Arveson) are very natural:

- Suppose that X and Y are vector spaces and that $T\colon X \to Y$ is linear. If each of the matrix spaces $M_n(X)$ and $M_n(Y)$ have a norm (written $\|\cdot\|_n$), then we say that T is *completely isometric*, or is a *complete isometry*, if

$$\|[T(x_{ij})]\|_n = \|[x_{ij}]\|_n\,, \qquad \forall n \in \mathbb{N}, [x_{ij}] \in M_n(X)\,.$$

  Compare this to the definition of an isometry on page 13. Similarly, T is a *complete contraction* if

$$\|[T(x_{ij})]\|_n \leq \|[x_{ij}]\|_n\,, \qquad \forall n \in \mathbb{N}, [x_{ij}] \in M_n(X)\,.$$

  Compare this to the definition of a contraction on page 13. Finally, T is *completely bounded* if

$$\|T\|_{\rm cb} \stackrel{def}{=} \sup\big\{\|[T(x_{ij})]\|_n \,:\, n \in \mathbb{N}, [x_{ij}] \in {\rm Ball}(M_n(X))\big\} < \infty.$$

  Compare this to the definition of T on page 11. You will see that a complete isometry is an isometry (but not vice versa), a complete contraction is a contraction, and a completely bounded function is bounded, in fact with $\|T\| \leq \|T\|_{\rm cb}$ (this follows if you restrict the supremum in the last definition to the case that $n = 1$, this gives a smaller number).

- Exercise: If $S, T \in B(H)$ with $\|S\| \leq 1$ and $\|T\| \leq 1$, show that the function $x \mapsto SxT$ is a complete contraction on $B(H)$.

- We often think of two operator spaces X and Y as being the same if they are *completely isometrically isomorphic*, that is, if there exists a linear complete isometry from X onto Y. In this case we often write '$X \cong Y$ completely isometrically'.

  Example: consider the operator spaces R_n and C_n a few paragraphs above. The 'identity function', namely the transpose, from C_n to R_n is an isometry, as we remarked there. Call this function T. It is not a complete isometry: for example, above we found a matrix $[x_{ij}] \in M_2(C_2)$ with $\|[T(x_{ij})]\|_n \neq \|[x_{ij}]\|_2$.

  In fact it is possible to show that there does not exist any linear complete isometry from C_2 to R_2.

- At this point, we will need to recall that we proved at the end of Chapter 2, certain properties satisfied by matrices of operators. Let me recall two of them, and give them names:

(R1) $\|axb\|_m \leq \|a\|\|x\|_m\|b\|$, for all $m \in \mathbb{N}$ and all $a, b \in M_m$, and

(R2)

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}.$$

Since these hold whenever $x \in M_m(B(H))$ and $y \in M_n(B(H))$, for any Hilbert space H, they hold in particular whenever $x \in M_m(X)$ and $y \in M_n(X)$, for any operator space $X \subset B(H)$.

Conditions (R1) and (R2) above are often called *Ruan's axioms*. *Ruan's theorem* asserts that (R1) and (R2) actually *characterize* operator spaces. This result is fundamental to the subject in many ways. for example, it is used frequently to check that certain constructions which you can make with operator spaces, remain operator spaces.

Theorem 3.7.1 (Ruan) *Suppose that X is a vector space, and that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(X)$. Then X is completely isometrically isomorphic to a linear subspace of $B(H)$, for some Hilbert space H, if and only if conditions* (R1) *and* (R2) *above hold for all matrices $x \in M_m(X)$ and $y \in M_n(X)$.*

We will not prove this, it is quite lengthy.

- **The main point:** Just as normed spaces may be regarded as the pairs $(X, \|\cdot\|)$ consisting of a vector space and a norm on it, and these are 'exactly' the subspaces of commutative C^*-algebras (see the end of Section 2.2); so Ruan's theorem says that the pairs $(X, \{\|\cdot\|_n\})$ consisting of a vector space X and a norm on $M_n(X)$ for all $n \in \mathbb{N}$, which satisfies axioms (R1) and (R2), are 'exactly' the subspaces of $B(H)$ for Hilbert space H (or equivalently, by the Gelfand-Naimark theorem (Theorem 2.4.2), they are 'exactly' the subspaces of general C^*-algebras).

- Now we are ready to look at what the six steps should be in the 'quantization strategy' (see page 30-31), for 'quantizing' normed spaces. The idea is largely attributable to Effros. The first step we have already done: we observed earlier that the normed spaces were precisely the linear subspaces of the commutative C^*-algebras $C(K)$. The second step therefore is to remove the commutativity assumption; that is, we look at linear subspaces of general C^*-algebras. These are exactly the *operator spaces*, as we said in the last paragraph, which are nicely characterized by Ruan's theorem. Thus we regard the 'normed spaces' as the 'commutative operator spaces' (we will make this a little more precise in a few minutes), and conversely, we regard general operator spaces as 'noncommutative normed spaces'. This completes the second stage of the strategy. The third step in the strategy, is then to find good examples of operator spaces which occur naturally in mathematics and physics. We have seen some already: C^*-algebras, R_n and C_n. We will see more later. For a very nice, very rich, list of such examples, see Pisier's "Introduction to operator space theory" [4]. The fourth stage in the strategy, is to generalize the most important parts of the theory of normed spaces, to operator spaces. We will begin this process in the final lectures. Bear in mind though a principle I mentioned earlier; a 'good' theorem from the fourth stage, when applied to 'commutative operator spaces' (i.e. normed spaces) should give back a 'classical theorem'. The fifth step, studying truly noncommutative phenomena in operator space theory, we will not be able to reach. We will see a couple of Step 6 applications.

- Clearly, subspaces of operator spaces are again operator spaces.

- Any C^*-algebra A is an operator space. In fact, by the Gelfand-Naimark theorem 2.4.2, we may regard A as a $*$-subalgebra of $B(H)$. So A is a subspace of an operator space, and hence A is an operator space.

  It is not hard to write explicit formulae for the matrix norms $\|[x_{ij}]\|_n$ on $M_n(A)$ if A is a C^*-algebra. If $A = C(K)$ for a compact set K, then the formula is particularly nice:

  $$\|[f_{ij}]\|_n = \sup\big\{\|[f_{ij}(w)]\| \,:\, w \in K\big\} , \qquad [f_{ij}] \in M_n(C(K)).$$

  (Exercise: using Ruan's theorem, show that $C(K)$ with these matrix norms is an operator space.)

- Recall that if E is a normed space, then there is a canonical isometry $j : E \to C(K)$ where $K = \mathrm{Ball}(E^*)$. Since $C(K)$ is a C^*-algebra, it is an operator space, as we just said. By the formula in the last paragraph, it is not hard to see that the matrix norms of the operator space $C(K)$ induce, via j, the following matrix norms for E:

  $$\|[x_{ij}]\|_n = \sup\big\{\|[\varphi(x_{ij})]\| \,:\, \varphi \in \mathrm{Ball}(E^*)\big\} , \qquad [x_{ij}] \in M_n(E) .$$

  Every normed space may be canonically considered to be an operator space, and its matrix norms are the ones just described. Indeed, these are what one might call 'the commutative operator spaces'.

Chapter 4

Operator space theory and applications

In this chapter, we will begin the 'third step' in our 'quantization strategy', namely we will look at generalizations to operator spaces, of some basic results in functional analysis (for example the Hahn-Banach theorem, and the duality theory of normed spaces). We will also look at some other interesting facts. For example, we begin by a Step 6 item: we will show how the new theory can solve old problems.

4.1 An example of the use of operator spaces: the 'similarity problem'

If H is a Hilbert space, and if $T : H \to H$ is an operator, and if $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is a polynomial, then we know from linear algebra that by $p(T)$ we mean

$$p(T) \ = \ a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n \ .$$

We say that T is *polynomially bounded* if there is a constant $M > 0$ such that

$$\|p(T)\| \leq M \sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\}.$$

An example of a polynomially bounded operator is given by any contraction T (remember, 'contraction' means that $\|T\| \leq 1$). In fact von Neumann proved that for a contraction T,

$$\|p(T)\| \leq \sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\}.$$

The proof is not very hard, but we'll not prove it since my main purpose is to show the general ideas. We say that an operator $R : H \to H$ is *similar* to an operator $T : H \to H$, if there exists a bounded operator $S : H \to H$, which has an inverse S^{-1} which is also a bounded operator, such that

$$R \ = \ S^{-1} T S.$$

If this is the case, then notice that

$$R^k \;=\; (S^{-1}TS)(S^{-1}TS)\cdots(S^{-1}TS) = S^{-1}T^kS.$$

Thus if p is the polynomial above,

$$p(R) = a_0+a_1R+a_2R^2+\cdots+a_nR^n = a_0+a_1S^{-1}TS+a_2S^{-1}T^2S+\cdots+a_nS^{-1}T^nS = S^{-1}p(T)S.$$

It follows that if T is a contraction, then

$$\|p(R)\| = \|S^{-1}p(T)S\| \leq \|S^{-1}\|\|p(T)\|\|S\| \leq \|S^{-1}\|\|S\| \sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\},$$

using von Neumann's result. That is, R is polynomially bounded. We have proved:

Every operator similar to a contraction is polynomially bounded.

An obvious question, is if the reverse is true: is every polynomially bounded operator similar to a contraction? This resisted all attempts to prove it, and it became a major open problem in the subject of Operator Theory. It was solved quite recently by the French mathematician Gilles Pisier. His answer is NO. That is, he found a polynomially bounded operator which is not similar to a contraction. In fact, his answer shows how the 'operator spaces' we introduced in the last lecture, and their matrix norms, can be the key to a problem like the one above, a problem which on the face of it seems to have nothing to do with 'matrix normed vector spaces'.

I'd like to give an idea of the proof, explaining why it uses some of the ideas we have explored earlier together. Firstly, I will rephrase the definition of being 'polynomially bounded'. Let D be the set of complex numbers z with $|z| \leq 1$. We can regard any polynomial $p(z)$ as a continuous function from D to the scalars. We will work with $C(D)$, the set of all continuous functions from D to the scalars. Let A be the subspace of $C(D)$ consisting of the polynomials. Then for any polynomial p, we have

$$\sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\} \;=\; \|p\|_\infty,$$

in the notation on page 21. For an operator $T : H \to H$, let $\theta : A \to B(H)$ be the function $\theta(p(z)) = p(T)$. To say that T is polynomially bounded is exactly the same as saying that there is a $M > 0$ with

$$\|\theta(p(z))\| \leq M\|p\|_\infty,$$

which is exactly the definition of θ being bounded (see page 10).

Thus we can rephrase T being polynomially bounded, as θ being bounded. Notice that θ is a *homorphism*, that is $\theta(pq) = \theta(p)\theta(q)$ for two polynomials p and q (Exercize: check this.) In 1984, Paulsen proved the following theorem: If A is a subalgebra of a C^*-algebra, and if $f : A \to B(H)$ is a completely bounded (see the definition on page 35) homomorphism, then there is an invertible operator S in $B(H)$ such that the function $x \mapsto S^{-1}f(x)S$ is a completely contractive homomorphism. The converse is true too, but is pretty obvious.

Applying this result to our homomorphism θ above, gives quite easily that an operator $T \in B(H)$ is similar to a contraction if and only if θ is completely bounded (see the definition on page 35). Lets prove the important direction of this: If θ is completely bounded then by the last paragraph, there is an invertible S such that the function $x \mapsto S^{-1}\pi(x)S$ is contractive. Applying this function to the simplest polynomial $p(z) = z$ gives:

$$\|S^{-1}\pi(p)S\| = \|S^{-1}TS\| \leq \|p\|_\infty = 1.$$

Thus T is similar to a contraction.

Thus one can rephrase the open question mentioned above as asking whether θ being bounded implies θ is completely bounded. Or to find a counterexample, we need to find T such that θ is bounded, but not completely bounded.

You can see that we have reduced this open problem to an operator space problem, indeed a problem asking if the 'matrix norms' are necessary in a certain situation. The key point in Pisier's solution is to find 'rather big' matrix norms on the Hilbert space ℓ^2, so that ℓ^2 with these matrix norms is an operator space. The basic idea is something like: take the space C_n that we studied on page 34, and find other matrix norms on $M_m(C_n)$ which are 'different enough' from the usual ones, so that one can get something that is bounded, but not completely bounded.

4.2 Functions on operator spaces.

- Another way to rephrase Ruan's theorem is as follows: Let X be a vector space, *together with* a norm $\|\cdot\|_n$ on $M_n(X)$, for all $n \in \mathbb{N}$. Then the following are equivalent:

  (a) These norms satisfy conditions (R1) and (R2) from Chapter 3;

  (b) X is completely isometric to a linear subspace of $B(H)$ for a Hilbert space H.

  Such a space X, together with norms $\|\cdot\|_n$ on $M_n(X)$ satisfying these equivalent conditions above, is called an abstract operator space, or simply an operator space.

- The useful functions between operator spaces are the completely bounded linear functions, and in particular the complete contractions and complete isometries (defined on page 35 in Chapter 3). There are a few useful cases that completely bounded linear functions are just the same as the bounded linear functions. We will discuss two such cases:

  Proposition 4.2.1 *For a homomorphism f between C^*-algebras, the following are equivalent:* (i) *f is contractive,* (ii) *f is completely contractive,* (iii) *f is a $*$-homomorphism. In this case f is one-to-one if and only if it is completely isometric.*

  Proof: The main step is to use Proposition 2.4.1 which says that if f is a $*$-homomorphism then it is contractive. But the function $[x_{ij}] \mapsto [f(x_{ij})]$ is also easy to see is a $*$-homomorphism. So it is contractive too. Thus f is completely contractive. We omit the rest of the proof, most of which is similar to the idea we just used. $\square$

The next result is of a similar flavor:

Proposition 4.2.2 *If X is an operator space, and if $f \in X^*$, then $\|f\| = \|f\|_{cb}$. This is also true for linear functions $f : X \to C(K)$, for a compact space K.*

Proof: If $[a_{ij}] \in M_n$ then we showed at the end of Section 1.4 that

$$\|[a_{ij}]\| \;=\; \sup \left\{ \left|\sum_{ij} a_{ij} z_j \overline{w_i}\right| \;:\; z = [z_j],\; w = [w_i] \in \text{Ball}(\mathbb{R}^n)\right\}.$$

Now let $f \in X^*$. It is clear that $\|f\| \leq \|f\|_{cb}$, by considering the case $n = 1$ in the definition of $\|f\|_{cb}$ in Chapter 3. To prove that $\|f\|_{cb} \leq \|f\|$, choose a matrix $[x_{ij}] \in M_n(X)$. We need to show that $\|[f(x_{ij})]\| \leq \|f\|\|[x_{ij}]\|_n$, we'd be done, by the definition of $\|f\|_{cb}$ in Chapter 3. To do this, we use the formula at the start of the present proof:

$$\|[f(x_{ij})]\| = \sup\left\{ \left|\sum_{ij} f(x_{ij}) z_j \overline{w_i}\right| \;:\; z = [z_j],\; w = [w_i] \in \text{Ball}(\mathbb{R}^n)\right\}.$$

But

$$\left|\sum_{ij} f(x_{ij}) z_j \overline{w_i}\right| \;=\; \left|f\Big(\sum_{ij} \overline{w_i} x_{ij} z_j\Big)\right| \leq \|f\| \left\|\sum_{ij} \overline{w_i} x_{ij} z_j\right\|.$$

Let a be the matrix which has the numbers $\overline{w_1}, \overline{w_2}, \cdots, \overline{w_n}$ on the first row, and otherwise is all zeros. Let b be the matrix which has the numbers $z_1, z_2, \cdots, z_n$ as the first column, and otherwise is all zeros. Then by facts on page 26, the norms of a and b are ≤ 1, if $z = [z_j],\; w = [w_i] \in \text{Ball}(\mathbb{R}^n)$. Also, $\sum_{ij} \overline{w_i} x_{ij} z_j$ is precisely the 1-1 entry of the matrix $a[x_{ij}]b$. By axioms (R1) and (R2),

$$\left\|\sum_{ij} \overline{w_i} x_{ij} z_j\right\| \;\leq\; \|a[x_{ij}]b\|_n \;\leq\; \|a\|\|[x_{ij}]\|_n\|b\| \leq \|[x_{ij}]\|_n.$$

Putting the facts above together,

$$\|[f(x_{ij})]\| \;\leq\; \|f\| \sup\left\{ \left\|\sum_{ij} \overline{w_i} x_{ij} z_j\right\| \;:\; z = [z_j],\, w = [w_i] \in \text{Ball}(\mathbb{R}^n)\right\} \;\leq\; \|f\|\|[x_{ij}]\|_n,$$

which is what we want.

Exercise: Prove the case that $f : X \to C(K)$, with the Hint: use the first part of the Lemma, applied to the functionals $x \mapsto f(x)(w)$ on X, for $f \in C(K), w \in K$. Also use a formula from page 37. $\square$

- We now state the generalization of the Hahn-Banach theorem to operator spaces:

Theorem 4.2.3 (Noncommutative Hahn-Banach theorem) *Suppose that Y is a subspace of an operator space X, and that u is a completely bounded linear function from Y into M_n (or into $B(H)$, where H is a Hilbert space). Then there exists a completely bounded linear function $\tilde{u}$ from X into M_n (or into $B(H)$) extending u (that is, $\tilde{u}(y) = u(y)$ for all $y \in Y$). Moreover this can be done with $\|\tilde{u}\|_{cb} = \|u\|_{cb}$.*

- We will not prove this theorem, again it is a bit lengthy. However let us check that it is a 'good noncommutative theorem'; that is, if you restrict it to 'commutative operator spaces' then you should get back the usual Hahn-Banach theorem. To see this, note that the 1×1 matrices are the scalars, so that any functional $f \in Y^*$ is a completely bounded linear function from Y into M_1, with $\|f\| = \|f\|_{cb}$, by Lemma 4.2.2. Thus, by the noncommutative Hahn-Banach theorem above, there is a completely bounded linear function $\tilde{f}$ from X into M_1 extending f, with $\|\tilde{f}\|_{cb} = \|f\|_{cb}$. By Lemma 4.2.2 again, $\|\tilde{f}\| = \|\tilde{f}\|_{cb} = \|f\|$. This gives the Hahn-Banach theorem from Chapter 2.

- Remember that we are trying to generalize the basic functional analysis that we met in Chapter 2. Now let us generalize the spaces $B(X, Y)$ of bounded linear operators between normed spaces X and Y. Remember that $B(X, Y)$ is a normed space again. So if X and Y are operator spaces, then we need to make $CB(X, Y)$, the space of completely bounded linear functions from X to Y, into an operator space too. By Ruan's theorem, this reduces to finding the right matrix norms. That is, the right norm on $M_n(CB(X, Y))$. We do this by defining, for $[u_{ij}] \in M_n(CB(X, Y))$, the norm $\|[u_{ij}]\|_n$ to be the completely bounded norm of the function $x \mapsto [u_{ij}(x)]$ from X to $M_n(Y)$. In other words,

$$\|[u_{ij}]\|_n = \sup\{\|[u_{ij}(x_{kl})]\|_{nm} : [x_{kl}] \in \text{Ball}(M_m(X)),\ m \in \mathbb{N}\}.$$

  Here the matrix $[u_{ij}(x_{kl})]$ is indexed on rows by i and k and on columns by j and l. Another way to describe what we have done, is to say that we are giving $M_n(CB(X, Y))$ the norm it must have in order for the canonical isomorphism between $M_n(CB(X, Y))$ and $CB(X, M_n(Y))$ (namely the function taking a matrix $[u_{ij}]$ whose entries are functions from X to Y, to the function from X to $M_n(Y)$ given by $x \mapsto [u_{ij}(x)]$) to be an isometry. That is,

$$M_n(CB(X, Y)) \cong CB(X, M_n(Y)) \quad \text{isometrically.}$$

  One may see that $CB(X, Y)$ is now an operator space by appealing to Ruan's theorem 3.7.1. There is also a direct way to see it without using Ruan's theorem.

4.3 Duality of operator spaces

The special case of $CB(X, Y)$ when $Y = \mathbb{C}$ is particularly important. In this case, for any operator space X, we see that $CB(X, \mathbb{C})$ is again an operator space. The latter space equals

$X^* = B(X, \mathbb{C})$, by Lemma 4.2.2. We call X^*, viewed as an operator space in this way, the *operator space dual* of X. By the last centered equation above (with $Y = \mathbb{C}$),

$$M_n(X^*) \cong CB(X, M_n) \quad \text{isometrically.}$$

Again the isomorphism here is the function taking a matrix $[f_{ij}]$ whose entries are functions from X to $\mathbb{C}$, to the function from X to M_n given by $x \mapsto [f_{ij}(x)]$.

An operator space Y is said to be a *dual operator space* if Y is completely isometrically isomorphic to the operator space dual X^* of an operator space X (defined in the last paragraph). We also say that X is an *operator space predual* of Y, and sometimes we write X as Y_*.

Unless otherwise indicated, in what follows the symbol X^* denotes the operator space dual. Since X^* is an operator space, $X^{**} = (X^*)^*$ is again an operator space. Remember from Section 2.1 that there is a function $i_X : X \to X^{**}$ defined by $i_X(x)(f) = f(x)$, for $f \in X^*, x \in X$. We showed there that i_X is an isometry. In the operator space case, we need to know that i_X is a complete isometry, so that we can view X as a subspace of its 'second dual':

Theorem 4.3.1 (Blecher-Effros-Paulsen-Ruan) *If X is an operator space then $X \subset X^{**}$ completely isometrically via the canonical function i_X defined above.*

Proof: We can suppose that X is a subspace of $B(H)$, for a Hilbert space H. Fix $n \in \mathbb{N}$ and $[x_{ij}] \in M_n(X)$. We first show that $\|[i_X(x_{ij})]\|_n \leq \|[x_{ij}]\|_n$. By the second last centered equation in the last section, the norm $\|[i_X(x_{ij})]\|_n$ in $M_n((X^*)^*)$ is

$$\begin{aligned} \|[i_X(x_{ij})]\|_n &= \sup\big\{\|[i_X(x_{ij})(f_{kl})]\|_{nm} : [f_{kl}] \in \mathrm{Ball}(M_m(X^*)),\, m \in \mathbb{N}\big\} \\ &= \sup\big\{\|[f_{kl}(x_{ij})]\|_{nm} : [f_{kl}] \in \mathrm{Ball}(M_m(X^*)),\, m \in \mathbb{N}\big\} \\ &\leq \|[x_{ij}]\|_n, \end{aligned}$$

the last line because of the second last centered equation at the end of the last section, telling us what it means for $[f_{kl}] \in \mathrm{Ball}(M_m(X^*))$; that is, for $[f_{kl}]$ to have norm ≤ 1 in $M_m(X^*)$.

To see that i_X is completely isometric, it remains to show the other direction $\|[i_X(x_{ij})]\|_n \geq \|[x_{ij}]\|_n$. Because $\|[i_X(x_{ij})]\|_n$ is given by the supremum in the last paragraph, it suffices to show that given $\epsilon > 0$. there exists $[f_{kl}] \in M_m(X^*)$ of norm ≤ 1, such that

$$\|[f_{kl}(x_{ij})]\|_n \geq \|[x_{ij}]\|_n - \epsilon.$$

By the last centered equation at the end of the last section, if we write u for the function $u(x) = [f_{kl}(x)]$ on X then u is a complete contraction if and only if $[f_{kl}] \in M_m(X^*)$ has norm ≤ 1. Thus to achieve our goal, it suffices to find for a given $\epsilon > 0$, an integer m and a complete contraction $u \colon B(H) \to M_m$ such that $\|[u(x_{ij})]\|_n \geq \|[x_{ij}]\| - \epsilon$. Now $[x_{ij}] \in M_n(X) \subset M_n(B(H)) \cong B(H^{(n)})$ (see end of Chapter 1). Thus $[x_{ij}]$ may be interpreted as an operator from the Hilbert space $H^{(n)}$ to itself. Now the norm of any operator $T \in B(K)$, for any Hilbert space K, is given by the formula

$$\|T\| = \sup\{|\langle Ty, z\rangle| : y, z \in \mathrm{Ball}(K)\},$$

as may be proved by the argument at the end of Section 1.4. In our case,

$$\|[x_{ij}]\|_n = \sup\{|\langle [x_{ij}]y, z\rangle| : y, z \in \text{Ball}(H^{(n)})\}.$$

By a basic property of supremums, if $\epsilon > 0$ is given, there exists $y, z \in \text{Ball}(H^{(n)})$ such that $|\langle [x_{ij}]y, z\rangle| > \|[x_{ij}]\|_n - \epsilon$. We can write y as a column consisting of n vectors $\zeta_1, \zeta_2, \cdots, \zeta_n$, each in H. Similarly, z is a column consisting of $\eta_1, \eta_2, \cdots, \eta_n \in H$. We have $\sqrt{\sum_l \|\zeta_l\|^2} = \|y\| \leq 1$, and similarly $\sqrt{\sum_k \|\eta_k\|^2} \leq 1$. Then $\langle [x_{ij}]y, z\rangle = \sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle$, and so We have

$$\Big|\sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle\Big| \geq \|[x_{ij}]\| - \epsilon.$$

Let $K = \text{Span}\,\{\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n\}$, the span in H. This is finite dimensional, and it is an inner product space since it is a subspace of an inner product space. Thus by a fact stated at the end of Section 1.5, there is a $*$-isomorphism $\pi : B(K) \to M_m$, where $m = \dim(K)$. We know π is a complete isometry by Proposition 4.2.1. There is a well-known fact about Hilbert spaces, which you may have proved a variant of in a linear algebra class, that for any subspace K of a Hilbert space H, there is a 'projection' P_K of norm 1 from H onto K. This means that $P_K(x) = x$ if $x \in K$. Let $T\colon B(H) \to B(K)$ be the function $T(x) = P_K x \epsilon_K$, where $\epsilon_K : K \to H$ is just the 'inclusion function' $\epsilon_K(x) = x$ if $x \in K$. Let $u = \pi \circ T$. Then T is completely contractive (by the Exercise on page 36). As in the last paragraph,

$$\|[T(x_{ij})]\|_n = \sup\{|\langle [T(x_{ij})]y', z'\rangle| : y', z' \in \text{Ball}(K^{(n)})\},$$

and we can write $\langle [T(x_{ij})]y, z\rangle = \sum_{i,j} \langle T(x_{ij})\zeta_j, \eta_i\rangle$. This implies that

$$\|[T(x_{ij})]\|_n \geq \Big|\sum_{i,j} \langle T(x_{ij})\zeta_j, \eta_i\rangle\Big| = \Big|\sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle\Big|.$$

Thus, if we set $u = \pi \circ T$, then u is completely contractive (why?), and

$$\|[u(x_{ij})]\|_n = \|[\pi(T(x_{ij}))]\|_n = \|[T(x_{ij})]\|_n \geq \Big|\sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle\Big| \geq \|[x_{ij}]\| - \epsilon,$$

using the fact at the end of the last paragraph. This is the desired inequality. ☐

One can show, for example, that $R_n^* = C_n$ and $C_n^* = R_n$.

Other aspects of the duality of normed spaces also go through beautifully. See the texts in our reference list for more on this.

4.4 The abstract characterization of operator algebras

Good examples of operator spaces include *operator algebras*, that is, subalgebras of $B(H)$ for a Hilbert space H. Another triumph for operator space theory was that it gave new results in the theory of operator algebras, as we will see.

Note that R_n and C_n are subalgebras of M_n, so they are operator algebras (but the product on these spaces is not so interesting, for example there is no 1 for this product. A better example is the upper triangular matrices:

$$\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \cdot & \cdot & \cdots & \cdot & \\ \cdot & \cdot & \cdots & \cdot & \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}.$$

(Check that the product of two such matrices is again upper triangular.)

If A is an operator algebra, a subalgebra of $B(H)$, then $M_n(A)$ is a subalgebra of $M_n(B(H))$. Remember that $M_n(B(H))$ can be identified with $B(H^{(n)})$. Since the norm of operators has the property that $\|ST\| \leq \|S\|\|T\|$ (see page 12), we conclude that $M_n(A)$ has the property that

$$\|ab\| \leq \|a\|\|b\|,$$

for any $n \in \mathbb{N}$ and any matrices $a, b \in M_n(A)$. This proves the easy direction of the following result:

Theorem 4.4.1 (Blecher-Ruan-Sinclair) *An operator space A with a product (that is, a multiplication $A \times A \to A$), is an operator algebra if and only if*

$$\|ab\| \leq \|a\|\|b\|,$$

for any $n \in \mathbb{N}$ and any matrices $a, b \in M_n(A)$.

We will sketch a proof of the difficult direction later if we have time. Such a theorem (which if you like, is the 'algebra version' of Ruan's theorem), is very useful in practice, because the conditions are quite easy to verify. It allows one to develop a *general theory* of operator algebras, i.e. a general theory of algebras consisting of operators on a Hilbert space. See e.g. [1].

4.5 Banach-Stone theorems and the noncommutative Shilov boundary

A beautiful result due to Banach and Stone says that for two compact topological spaces K_1 and K_2, the following are equivalent:

(a) K_1 and K_2 are homeomorphic (i.e. they are 'equal' as topological spaces),

(b) The algebras $C(K_1)$ and $C(K_2)$ are isomorphic as algebras,

(c) The normed spaces $C(K_1)$ and $C(K_2)$ are linearly isometric (i.e. they are 'equal' as normed spaces).

It is quite easy to prove the equivalence of (a) and (b), and that (b) implies (c). That (c) implies (b) is the hard direction. It can be seen quite easily though from a construction called the 'Shilov boundary'. Before we discuss the Shilov boundary, I'll just mention some noncommutative versions of the Banach-Stone theorem. Kadison showed that two noncommutative C^*-algebras are 'Jordan-isomorphic' if and only if they are linearly isometric. In fact, in the operator space framework, the result becomes simpler: two C^*-algebras are $*$-isomorphic if and only if they are linearly completely isometric. Also, two general operator algebras which are linearly completely isometric, are also completely isometrically homomorphic.

This is saying that somehow the 'vector space structure' plus the 'matrix norms' determine the product (the multiplication) on an operator algebra. That is, suppose somebody gives you an operator algebra, but that you have forgotten what the product (the multiplication is). The point is that *if you know what the matrix norms are*, you can use them to find what the product was! This can be done explicitly using a theory I have introduced called *multipliers of operator spaces* (see e.g. [1]). I will not have time to discuss this theory much, but I originally developed it in terms of a construction called the 'noncommutative Shilov boundary', invented by Arveson and Hamana. I'd like to tell you a little about the noncommutative Shilov boundary, because it is a good illustration of why operator spaces are indeed very 'noncommutative'.

We will first talk about the 'classical' Shilov boundary. Remember that we said that every normed space is embeddable isometrically as a subspace of a $C(K)$ space. A good question is, what is the *smallest* compact space K that works? This 'smallest space' K is called the Shilov boundary, and written ∂X. It is very important in several branches of functional analysis, like 'Choquet theory' and the theory of 'function spaces' and 'function algebras', because it is a good tool to study the space X.

Now lets discuss the noncommutative version. Here we start with an operator space X and look for the 'smallest C^*-algebra A such that $X \subset A$ completely isometrically. We write this smallest C^*-algebra as ∂X.

Lets look at an interesting example. Take $\mathbb{R}^n$, but with its $\|\cdot\|_1$ norm:

$$\left\| \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \right\|_1 = \sum_k |z_k|.$$

If X is this normed space, one can show that $X = Y^*$ where Y is the commutative C^*-algebra $\ell^\infty(S)$ for an n point set $S = \{1, 2, \cdots, n\}$. Thus X gets an operator space structure (being the dual of a C^*-algebra). As a normed space, it is just $\mathbb{R}^n$ with the $\|\cdot\|_1$ norm. It turns out that its noncommutative Shilov boundary ∂X is a well known C^*-algebra called $C^*(F_n)$, the C^*-algebra of the free group with n generators. You dont need to know anything about this C^*-algebra, except that it is very well known, and very very noncommutative!! This is a typical phenomenon in operator space theory; when you get into it in detail, you find the calculations involve the high degree of 'noncommutativity' characteristic of 'noncommutative mathematics'.

By looking at these noncommutative Shilov boundaries, one can prove the noncommutative Banach-Stone theorem mentioned above. One can also define the 'multiplier algebras' of an operator space that I just mentioned. For example, if $1 \in X$ then the left multiplier algebra may be defined as $\{a \in \partial X : ax \in X \, \forall x \in X\}$ This is clearly an operator algebra, since it is clearly a subalgebra of the C^*-algebra ∂X. A very useful alternative formulation due to Blecher-Effros-Zarikian, is that the unit ball of this algebra corresponds exactly to the linear functions $T : X \to X$ such that

$$\left\| \begin{bmatrix} T(x) \\ y \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|.$$

In this way, we have associated an operator algebra to any operator space. This algebra can be very useful. For example, from the formulation in the last paragraph due to Blecher-Effros-Zarikian, it is very easy to prove the theorem of Blecher-Ruan-Sinclair on page 46 characterizing operator algebras. (Display slide with the proof).

For further reading on operator algebras, read [1]. You can download Chapter 1 from http://www.oup.co.uk/pdf/0-19-852659-8.pdf.

For further reading on operator spaces, including more historical references for facts above, consult the books listed in the Bibliography below.

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