Lecture 1
Operator spaces and their duality

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I. Introduction.

In noncommutative analysis we replace scalar valued functions by operators.

functions $\rightsquigarrow$ operators

In classical linear analysis one often solves a problem by working in a normed vector space of functions, using topology and measure and integration, and functional analytic tools

Eg. Choquet theory in the study of function algebras

When going NC, one hopes:

- C^*-algebra theory replaces topology

- Von Neumann algebra replaces arguments using measure and integrals

- ‘Operator space theory’ replaces Banach space techniques

Things can get scary ...

... e.g. replacing a classical argument with a dozen function inequalities, by operator inequalities

... but it is a land of miracles

In these talks, always:

H, K are Hilbert spaces

$B(H) = \{$ bounded linear $T : H \to H\}$

Operator space = closed subspace $X \subset B(H)$

In partic., X is a normed space

but it is more ...

A hidden piece of structure in $B(H)$:

A matrix of operators is an operator:

$$[T_{ij}] = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$$

in $M_n(B(H))$, may be viewed as an operator $H^{(n)} \to H^{(n)}$:

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \zeta_n \end{bmatrix} = \begin{bmatrix} \sum_k T_{1k}\zeta_k \\ \sum_k T_{2k}\zeta_k \\ \cdot \\ \cdot \\ \sum_k T_{nk}\zeta_k \end{bmatrix}.$$

Thus $[T_{ij}]$ **has a natural norm**:

$$\|[T_{ij}]\|_n = \sup\{\|[T_{ij}]\vec{\zeta}\| : \vec{\zeta} \in K^{(n)}, \|\vec{\zeta}\| \leq 1\}.$$

This phenomena, that a matrix of operators is an operator, is fairly ubiquitous in 'Operator Algebra'. It shows up almost everywhere, such as in 'stabilization', or when you want to do tensor products of operator algebras. And this is the main reason why there has been so little overlap between Banach spaces and Operator Algebra: You cannot even talk about some of the basic operator algebra constructions, such as the tensor product, in the Banach category. Need the functional analysis to reflect this matrix phenomena.

Crucial formula:

$$M_n(B(H)) \cong B(H^{(n)}) \text{isometrically}$$

If $X \subset B(H)$ we get an inherited norm on

$$M_n(X) \subset M_n(B(H)) \cong B(H^{(n)})$$

Note: if A is a C^*-subalgebra of $B(H)$ then these norms on $M_n(A)$ are just the usual C^*-algebra norm there

A *matrix normed space* is a pair

$$(X , \{\| \cdot \|_n\}_{n=1,2,\cdots})$$

where X is a vector space, and $\| \cdot \|_n$ is a norm on $M_n(X)$.

Completely bounded linear maps $T : X \to Y$:

$$\|T\|_{cb} = \sup\{\|[T(x_{ij})]\|_n : n \in \mathbb{N}, \|[x_{ij}]\|_n \leq 1\} < \infty$$

Complete isometry:

$$\|[T(x_{ij})]\|_n = \|[x_{ij}]\|_n \quad , \quad \text{for all} \quad n \in \mathbb{N}, x_{ij} \in X \quad .$$

Complete contraction: $\|T\|_{cb} \leq 1$.

It is easy to see that $B(H)$, with the natural norm $\|\cdot\|_n$ on $M_n(B(H))$ satisfies the following, for all $x \in M_n(X), y \in M_m(X)$, and scalar matrices α, β:

$$(R1) \quad \left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$$

$$(R2) \quad \|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| \quad .$$

Ruan's theorem. A matrix normed space satisfies (R1) and (R2) if and only if it is completely isometric to a subspace of $B(H)$, for some Hilbert space H.

The new category: *operator spaces* = matrix norm spaces as in the theorem.

The morphisms between operator spaces .... $CB(X, Y)$.

Summary: Operator spaces are just the linear subspaces of $B(H)$, with matrix norms... and we have a nice way of viewing them completely abstractly.

Are the matrix norms a burden ... ?

Not usually. Usually in the middle of a proof it is enough to check that things work for the first norm $\| \cdot \|_1$, and say "...similarly for the $\| \cdot \|_n$ norm for $n \geq 2$.

An application of Ruan's theorem: Quotients

If E is a subspace of an vector space X, then $M_n(X/E) \cong M_n(X)/M_n(E)$ linearly

If X is an operator space and E is closed, we can use this formula to give $M_n(X/E)$ a norm:

$$\|[x_{ij} + E]\|_n = \inf\{\|[x_{ij} + y_{ij}]\|_n : y_{ij} \in E\}.$$

The quotient map $q : X \to X/E$ is completely contractive

Exercise: Use Ruan's theorem to show X/E is an operator space.

II. Duality of operator spaces

Duality in mathematics:

Object $X \rightsquigarrow$ Dual object $X^* \rightsquigarrow$ 2nd dual X^{**}

Sometimes $X^{**} \cong X$, usually $X \hookrightarrow X^{**}$

For example, normed linear spaces over field $\mathbb{F}$:

$$X \rightsquigarrow X^* = B(X, \mathbb{F}) \rightsquigarrow X^{**} = B(X^*, \mathbb{F})$$

Note that here it is crucial that X^* is in the same category as X

$B(X, Y)$ is a normed space with operator norm

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$$

Our topic in this lecture is:

Duality for spaces X of Hilbert space operators

What is the 'dual object' X^*?

Want it to be another operator space.

Q: Is there a canonical good way to do this?

A: Yes! B-Paulsen, Effros-Ruan, B $\sim$ 1990.

The conceptual difficulty:

Finite dimensional case: $B(H) = M_n$ with 'operator norm'

$$\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}$$

$$= \text{largest eigenvalue of } B$$

where $B = |A|$; that is, $B^2 = A^*A$.

The dual normed space S_n^1 is just M_n, but with 'trace class norm'

$$|||A||| = \text{sum of the eigenvalues of } B$$

This does not look like an operator norm!

The morphisms between operator spaces .... $CB(X, Y)$.

Define matrix norms on $CB(X, Y)$:

$$\|[T_{ij}]\|_n = \sup\{\|[T_{ij}(x_{kl})]\| : \|[x_{kl}]\|_m \leq 1\}$$

Then $CB(X, Y)$ is also an operator space!

A new duality:

$$X^* = CB(X, \mathbb{C})$$

The **operator space dual**

Note: $CB(X, \mathbb{C}) = B(X, \mathbb{C})$ isometrically (Exercise.)

Dual operator space = op. space Y which is completely isometrically isomorphic to the operator space dual X^* of an operator space X

Also say: X is an *operator space predual* of Y, and write $X = Y_*$.

Henceforth the symbol X^* denotes the operator space dual.

Since X^* is an operator space, $X^{**} = (X^*)^*$ is again an operator space.

Theorem: $\hat{}\ : X \to X^{**}$ completely isometrically.

Proof. Can suppose that X is a subspace of $B(H)$, for a Hilbert space H.

Fix $n \in \mathbb{N}$ and $[x_{ij}] \in M_n(X)$. We first show that $\|[i_X(x_{ij})]\|_n \leq \|[x_{ij}]\|_n$. By definition, the norm $\|[i_X(x_{ij})]\|_n$ in $M_n((X^*)^*)$ equals

$$\text{sup}\Big\{\|[i_X(x_{ij})(f_{kl})]\|_{nm} : [f_{kl}] \in \text{Ball}(M_m(X^*)),\, m \in$$
$$= \text{sup}\Big\{\|[f_{kl}(x_{ij})]\|_{nm} : [f_{kl}] \in \text{Ball}(M_m(X^*)),\, m \in \mathbb{N}$$
$$\leq \|[x_{ij}]\|_n,$$

the last line by definition of $[f_{kl}] \in \text{Ball}(M_m(X^*))$.

Next: the other direction $\|[i_X(x_{ij})]\|_n \geq \|[x_{ij}]\|_n$.

Because $\|[i_X(x_{ij})]\|_n$ is given by the supremum above, it suffices to show that given $\epsilon > 0$, there exists $[f_{kl}] \in M_m(X^*)$ of norm ≤ 1, such that

$$\|[f_{kl}(x_{ij})]\| \geq \|[x_{ij}]\|_n - \epsilon.$$

By definition of the matrix norms on X^*, if we write u for the function $u(x) = [f_{kl}(x)]$ on X then :

u is a complete contraction if and only if $[f_{kl}] \in M_m(X^*)$ has norm ≤ 1.

Thus it suffices to find for a given $\epsilon > 0$, an integer m and a complete contraction $u\colon B(H) \to M_m$ such that $\|[u(x_{ij})]\|_n \geq \|[x_{ij}]\| - \epsilon$.

Now $[x_{ij}] \in M_n(X) \subset M_n(B(H)) \cong B(H^{(n)})$
Thus $[x_{ij}]$ 'is' an operator on $H^{(n)}$

The norm of any operator $T \in B(K)$, for any Hilbert space K, is given by the formula

$$\|T\| = \text{sup}\{|\langle Ty, z\rangle| : y, z \in \text{Ball}(K)\}$$

In our case,

$$\|[x_{ij}]\|_n = \text{sup}\{|\langle [x_{ij}]y, z\rangle| : y, z \in \text{Ball}(H^{(n)})\}.$$

So, if $\epsilon > 0$ is given, there exists $y, z \in \text{Ball}(H^{(n)})$ such that $|\langle [x_{ij}]y, z\rangle| > \|[x_{ij}]\|_n - \epsilon$.

Write y as a column with entries $\zeta_1, \zeta_2, \cdots, \zeta_n \in H$. Similarly, z is a column of $\eta_1, \eta_2, \cdots, \eta_n \in H$. Then $\langle [x_{ij}]y, z\rangle = \sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle$, and so

$$\left|\sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle\right| \geq \|[x_{ij}]\| - \epsilon.$$

Let $K = \text{Span}\,\{\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n\}$ in H.

This is finite dimensional, so there is an isometric $*$-isomorphism $\pi : B(K) \to M_m$, where $m = \dim(K)$.

Let P_K be the projection from H onto K. Let $T\colon B(H) \to B(K)$ be the function $T(x) = P_K x_{|K}$. Let $u = \pi \circ T$.

Then T, π and u are completely contractive (Exercise).

Now $\langle [T(x_{ij})]y, z\rangle = \sum_{i,j}\langle T(x_{ij})\zeta_j, \eta_i\rangle$, and so

$$\|[T(x_{ij})]\|_n \geq \left|\sum_{i,j}\langle T(x_{ij})\zeta_j, \eta_i\rangle\right| = \left|\sum_{i,j}\langle P_K x_{ij}\zeta_j, \eta_i\rangle\right|$$

$$= \left|\sum_{i,j}\langle x_{ij}\zeta_j, \eta_i\rangle\right|,$$

the last step since $\eta_i \in K$. Thus, if we set $u = \pi \circ T$, then

$$\|[u(x_{ij})]\|_n = \|[\pi(T(x_{ij}))]\|_n = \|[T(x_{ij})]\|_n$$

$$\geq \left|\sum_{i,j}\langle x_{ij}\zeta_j, \eta_i\rangle\right| \geq \|[x_{ij}]\| - \epsilon,$$

using the fact at the end of the last paragraph. This is the desired inequality. $\square$

This is the ‘right’ duality ...

I’ll just give one piece of evidence:

Classical isometric identifications:

$$\ell^\infty_n \bar{\otimes} \ell^\infty_m = \ell^\infty_{mn} = B(\ell^1_n, \ell^\infty_m) = (\ell^1_n \widehat{\otimes} \ell^1_m)^*$$

What about $M_n \bar{\otimes} M_m = M_{mn}$? To make a similar statement in this case, MUST use the operator space dual, then

$$M_n \bar{\otimes} M_m = M_{mn} = CB(S^1_n, \ell^\infty_m) = (S^1_n \widehat{\otimes} S^1_m)^*$$

(B-Paulsen, generalized to von Neumann algebras by Effros-Ruan)

One needs the operator space dual to do most things in the field of operator spaces.

E.g. See books of Effros-Ruan, Pisier, Paulsen, B-Le Merdy ’00-’04.

Recall $B(H) = S_1(H)^*$ where $S^1(H)$ is the *trace class*

Thus $S^1(H) \subset B(H)^*$. Since $B(H)^*$ is an operator space, $S^1(H)$ becomes an operator space.

Prop. $S^1(H)^* \cong B(H)$ compl. isometrically

Proof. The usual map $B(H) \to S^1(H)^*$ is an isometric c. contraction clearly. A moments thought shows that it is a compl. isometry because the map we called $u : B(H) \to M_n$ in the proof of the last theorem is weak* continuous. $\square$

If $u : X \to Y$ is a map between operator spaces, then $u^* : Y^* \to X^*$ satisfies $\|u^*\|_{cb} = \|u\|_{cb}$. (Exercise)

The usual properties of Banach space duality work for operator spaces.

Eg. $(X/E)^* \cong E^\perp$ if $E \subset X$

Lemma Any weak* closed subspace X of $B(H)$ is a dual operator space. Indeed, $(S^1(H)/X_\perp)^* \cong X$ completely isometrically.

Proof We have $(S^1(H)/X_\perp)^* \cong (X_\perp)^\perp$ c.i. in $S^1(H)^*$.

But $S^1(H)^* = B(H)$ c.i. and $(X_\perp)^\perp = X$. $\Box$

The converse is true too, so that 'dual operator spaces', and the weak* closed subspaces of some $B(H)$, are the same thing.

Lemma (Effros & Ruan) Any dual operator space is c. isometrically isomorphic, via a weak* homeomorphism, to a weak* closed subspace of $B(H)$, for some Hilbert space H.

Proof. (B) Suppose that $W = X^*$, let $I = \cup_n \text{Ball}(M_n(X))$, and for $x \in \text{Ball}(M_m(X)) \subset I$ set $n_x = m$. Define

$$J \,:\, W \longrightarrow \oplus_{x\in I}^{\infty} M_{n_x}$$

by $J(w) = ([\langle w, x_{ij}\rangle])_x$ in $\oplus_x M_{n_x}$.

Since the maps $w \mapsto \langle w, x_{ij} \rangle$ are w^*-continuous, and since $\oplus_x^{\text{fin}} M_{n_x}^*$ is dense in the Banach space predual $\oplus_x^1 M_{n_x}^*$ of $\oplus_x M_{n_x}$, it is easy to see that J is w^*-continuous too.

Thus by Krein-Smulian theorem, W is completely isometrically and w^*-homeomorphically isomorphic to a w^*-closed subspace of $\oplus_x M_{n_x}$.

The latter is a von Neumann subalgebra of some $B(H)$. $\square$

Some subtleties: Eg. X an operator space, which is also a dual normed space $X = Y^*$. Is X a dual operator space?

Not necessarily! (Le Merdy)

Note that there is only one possible reasonable way to make Y an operator space, because $Y \subset Y^{**} = X^*$.

We are usually only interested in 'good' preduals Y

'good': if you give Y the operator space structure just discussed, then $X = Y^*$ completely isometrically.

Fact: the predual of a von Neumann algebra is always 'good' (we showed this for $B(H)$).

Here is a simple example of an operator space with no 'good' Banach space preduals:

Let $B = B(H)$ with usual matrix norms.
Let $X = B(H)$ but with matrix norms

$$|||[x_{ij}]|||_n = \max\{\|[x_{ij}]\|_n, \|[q(x_{ji})]\|_n\}$$

where $q : B \to B/K(H)$ is the canonical map.

As a Banach space, $X = B$, and has a unique Banach space predual $S^1(H)$.

Note: $|||\cdot|||_n$ restricted to the copy of $M_n(K(H))$ is just the usual norm.

Thus if $Y = S^1(H)$ with the matrix norms coming from its duality with $(X, \{|||\cdot|||_n\})$:

$$\|Id \ : \ Y \ \longrightarrow \ S^1(H)\|_{cb} \ \leq \ 1.$$

Dualizing,

$$\|Id \ : \ B(H) \ \longrightarrow \ Y^*\|_{cb} \ \leq \ 1.$$

Thus if $Y^* = X$ completely isometrically, then since clearly

$$\|Id \ : \ X \ \longrightarrow \ B(H)\|_{cb} \ \leq \ 1,$$

we have $X = B(H)$ completely isometrically, a contradiction!!

This is a contradiction since $|||\cdot|||_n$ is not the usual matrix norms on $B(H)$. For if it was, then this immediately implies that for any $x_{ij} \in B(H), k_{ij} \in K(H)$ we have

$$\|[q(x_{ji})]\|_n \leq \|[x_{ij} + k_{ij}]\|_n$$

This means that

$$\|[q(x_{ji})]\|_n \leq \|[q(x_{ij})]\|_n$$

But the only C*-algebras with $\|[a_{ji})]\|_n = \|[a_{ij}]$ are commutative ones (this follows immediately from a result in the last part of this lecture, and the Calkin algebra is not commutative!

If A is a C^*-algebra then A^{**} has two canonical operator space structures.

The first is its 'operator space dual' matrix norms. The second are those arising from the fact that A^{**} is a C^*-algebra (any C^*-algebra has a canonical operator space structure).

Fact: These two operator space structures are the same.

A more general fact: $M_n(X)^{**} \cong M_n(X^{**})$ completely isometrically, for any operator space X. (But we shall not really need this.)

We also note that $M_n(X^*)$ can be shown to be a dual operator space, and Effros and Ruan observed that ...

if (x_t) is a bounded net in $M_n(Y^*)$, then $x_t \to x$ in the weak* topology if and only if each entry in x_t converges in the weak* topology in X to the corresponding entry in x.

III. Minimal operator spaces

If X is a Banach space, we make it into an operator space as follows: $MIN(X) = X$ but with matrix norms

$$\|[x_{ij}]\|_n = \sup\{\|[f(x_{ij})]\| : f \in Ball(X^*)\} \ .$$

This is called "MIN" because these are the smallest matrix norms making X into an operator space.

What we are really doing: Any Banach space $X \subset C(K)$ isometrically, where $K =$ Ball(X^*). The matrix norm above, is just the C^*-algebra norm on $M_n(C(K))$.

The assignment $X \rightsquigarrow MIN(X)$ embeds the category of Banach spaces in the category of operator spaces.

Easy fact: $\|T\| = \|T\|_{cb}$ for any bounded map $T : Y \to MIN(X)$, any operator space Y (Exercise).

IV. Review of complete positivity

An operator space X is *unital* if it has a distinguished element e, s.t. $\exists$ c. isometry $u\colon X \to A$ into a unital C^*-algebra with $u(e) = 1$.

An *operator system* $\mathcal{S}$ is a subspace of A containing 1 and is *selfadjoint* (i.e. $x^* \in \mathcal{S}$ iff $x \in \mathcal{S}$).

Write $\mathcal{S}_+ = \mathcal{S} \cap A_+$.

Say $u : \mathcal{S} \to \mathcal{S}'$ between operator systems is *positive* if $u(\mathcal{S}_+) \subset \mathcal{S}'_+$.

Say u is *completely positive* if $u_n\colon M_n(\mathcal{S}) \to M_n(\mathcal{S}')$ is positive for all $n \in \mathbb{N}$.

If $x \in B(K, H)$, then (exercise):

$$\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad \|x\| \leq 1.$$

Here '≥ 0' means 'positive in $B(H \oplus K)$'.

Because of this simple fact, it is easy to see that a unital map between operator systems is completely positive if and only if it is completely contractive.

Theorem (Stinespring) Let A be a unital C^*-algebra. A linear map $u\colon A \to B(H)$ is completely positive and unital if and only if there is a Hilbert space K, a unital $*$-homomorphism $\pi\colon A \to B(K)$, and an isometry $V: H \to K$ such that $u(a) = V^*\pi(a)V$ for all $a \in A$. This can be accomplished with $\|u\|_{\text{cb}} = \|V\|^2$.

Proposition (A Kadison–Schwarz inequality) If $u\colon A \to B$ is a unital completely positive linear map between unital C^*-algebras, then $u(a)^*u(a) \leq u(a^*a)$, for all $a \in A$.

Proof By Stinespring, $u = V^*\pi(\cdot)V$, with V an isometry and π a $*$-homomorphism. Thus

$$u(a)^*u(a) = V^*\pi(a)^*VV^*\pi(a)V$$

$$\leq V^*\pi(a)^*\pi(a)V = u(a^*a).$$

$\square$

Proposition Let $u\colon A \to B$ be a completely isometric unital surjection between unital C^*-algebras. Then u is a $*$-isomorphism.

Proof. By last result applied to both u and u^{-1} we have $u(x)^*u(x) = u(x^*x)$ for all $x \in A$. Now use the polarization identity. $\square$

Proposition Let $u\colon A \to B$ be as above. Suppose that $c \in A$ with $u(c)^* u(c) = u(c^* c)$. Then $u(ac) = u(a)u(c)$ for all $a \in A$.

An immediate consequence: Suppose that $u\colon A \to B$ is as above, and that there is a C^*-subalgebra C of A with $1_A \in C$, such that $\pi = u_{|C}$ is a $*$-homomorphism. Then

$$u(ac) = u(a)\pi(c) \quad \text{and} \quad u(ca) = \pi(c)u(a)$$

for $a \in A, c \in C$.

Theorem (Choi and Effros) Suppose that A is a unital C^*-algebra, and that $\Phi\colon A \to A$ is a unital, completely positive map with $\Phi \circ \Phi = \Phi$. Then:

(1) $R = \mathrm{Ran}(\Phi)$ is a C^*-algebra but with new product $r_1 \circ_\Phi r_2 = \Phi(r_1 r_2)$.

(2) $\Phi(ar) = \Phi(\Phi(a)r)$ and $\Phi(ra) = \Phi(r\Phi(a))$, for $r \in R$ and $a \in A$.

(3) If B is the C^*-subalgebra of A generated by R, and if R is given the product $\circ_\Phi$, then $\Phi_{|B}$ is a $*$-homomorphism from B onto R.

The proof is not hard.

If X is a subspace of $B(H)$, we define the *Paulsen system* to be the operator system

$$\mathcal{S}(X) = \begin{bmatrix} \mathbb{C}I_H & X \\ X^\star & \mathbb{C}I_H \end{bmatrix} = \left\{ \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} : x, y \in X, \ \lambda, \mu$$

in $M_2(B(H))$. The following shows that the operator system $\mathcal{S}(X)$ only depends on the operator space structure of X, and not on its representation on H.

Lemma (Paulsen) Suppose that for $i = 1, 2$, we are given Hilbert spaces H_1, H_2, and linear subspaces $X_1 \subset B(H_1)$ and $X_2 \subset B(H_2)$. Suppose that $u\colon X_1 \to X_2$ is a linear map. Let $\mathcal{S}_i$ be the following operator system inside $B(H_i \oplus H_i)$:

$$\mathcal{S}_i = \left[\begin{array}{cc} \mathbb{C}I_{H_i} & X_i \\ X_i^\star & \mathbb{C}I_{H_i} \end{array} \right] .$$

If u is contractive (resp. completely contractive, completely isometric), then

$$\Theta : \left[\begin{array}{cc} \lambda & x \\ y^* & \mu \end{array} \right] \mapsto \left[\begin{array}{cc} \lambda & u(x) \\ u(y)^* & \mu \end{array} \right]$$

is positive (resp. completely positive and completely contractive, a complete order injection) as a map from $\mathcal{S}_1$ to $\mathcal{S}_2$.

The proof is not hard.